Your TA: ____________ Seat #: [ ] – [ ]

Math 105 TOPICS IN MATHEMATICS
MIDTERM EXAM – I (Take-home)

March 6 (Fri), 2015

Instructor: Yasuyuki Kachi

Line #: 52920.

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This take-home part of Midterm Exam is worth 60 points and <u>is due in class Wednesday, March 11th, 2015. Submission after 1:00 pm, March 11th will not be accepted.</u>

- **Be sure to write your answers neatly, precisely, and with complete sentences. You may use notes and handed out materials, but no outside help.**
- **<u>Print off one entire set of this exam. Write answers in the printed sheets. You may not supply your own (blank) sheet.</u>**

⋆ In problem [I] below we work on a model where one can divide any dollar amount by any large number (integer). Also, we never round figures. So, one-third of a dollar is never the same as 33 cents (because 33 cents is one-third of 99 cents).

[I] (Take-home; 20pts) You open a bank account, deposit a dollar in that account.

(1) Your bank offers 100 percent interest annually.

After one year, your balance is $ ________ .

(2) Suppose your bank offers a compound interest with 100 percent rate annually.

(2a) After <u>two</u> years, your balance is $ ________ .

(2b) After <u>three</u> years, your balance is $ ________ .

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([I] continued)

(3) Suppose the compounding takes place five times annually. So every $\frac{1}{5}$-th of a year $\frac{1}{5} \cdot 100 = 20$ percent of your balance will be accrued as an interest.

(3a) After the $\frac{1}{5}$-th of a year, your balance is $ __________ .

(3b) After the $\frac{2}{5}$-th of a year, your balance is $ __________ .

(3c) After the one year, your balance is $\$ \left(1 + \frac{1}{\boxed{5}}\right)^{\Box}$.

(4) Suppose the compounding takes place 10^{100} times annually. So every $\frac{1}{10^{100}}$-th of a year, $\frac{1}{10^{100}}$ times 100 percent of your balance will be accrued as an interest.

After one year, your balance is $\$ \left(1 + \frac{1}{\Box}\right)^{\Box}$.

(5) Is your answer in (4)

☐ between \$1 and \$2. ☐ between \$2 and \$2.50.

☐ between \$2.50 and \$3.00. ☐ more than \$3.

(Check one.)

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[II] (Take-home; 20pts) (a) Use calculator to pull the decimal expressions of the numbers in each of (a5) through (a10).

(a1) $1+\dfrac{1}{1!} = \boxed{2}\,.\,\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}$,

(a2) $1+\dfrac{1}{1!}+\dfrac{1}{2!} = \boxed{2}\,.\,\boxed{5}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}$,

(a3) $1+\dfrac{1}{1!}+\dfrac{1}{2!}+\dfrac{1}{3!} = \boxed{2}\,.\,\boxed{6}\boxed{6}\boxed{6}\boxed{6}\boxed{6}\boxed{6}\ \ldots$,

(a4) $1+\dfrac{1}{1!}+\dfrac{1}{2!}+\dfrac{1}{3!}+\dfrac{1}{4!} = \boxed{2}\,.\,\boxed{7}\boxed{0}\boxed{8}\boxed{3}\boxed{3}\boxed{3}\ \ldots$,

(a5) $1+\dfrac{1}{1!}+\dfrac{1}{2!}+\dfrac{1}{3!}+\dfrac{1}{4!}+\dfrac{1}{5!}$

$= \square\,.\,\square\square\square\square\square\square\ \ldots$,

(a6) $1+\dfrac{1}{1!}+\dfrac{1}{2!}+\dfrac{1}{3!}+\dfrac{1}{4!}+\dfrac{1}{5!}+\dfrac{1}{6!}$

$= \square\,.\,\square\square\square\square\square\square\ \ldots$,

(a7) $1+\dfrac{1}{1!}+\dfrac{1}{2!}+\dfrac{1}{3!}+\dfrac{1}{4!}+\dfrac{1}{5!}+\dfrac{1}{6!}+\dfrac{1}{7!}$

$= \square\,.\,\square\square\square\square\square\square\ \ldots$,

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([II] continued)

(a8) $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!}$

$$= \square \,.\, \square\square\square\square\square\square \ \cdots \ ,$$

(a9) $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!}$

$$+ \frac{1}{9!} = \square \,.\, \square\square\square\square\square\square \ \cdots \ ,$$

(a10) $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!}$

$$+ \frac{1}{9!} + \frac{1}{10!} = \square \,.\, \square\square\square\square\square\square \ \cdots \ .$$

(b) Use calculator to find the smallest positive integer n such that

$$\left(1 + \frac{1}{n}\right)^n$$

is bigger than the value in (a4) above $\left(= 2.7083333...\right)$.

$$n = \underline{\qquad\qquad} .$$

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([II] continued)

(c) True or false :

"*Let* k *be an arbitrarily chosen positive integer, and fixed. If you choose a large enough* n, *then*

$$\left(1+\frac{1}{n}\right)^n \;>\; 1+\frac{1}{1\,!}+\frac{1}{2\,!}+\frac{1}{3\,!}+\frac{1}{4\,!}+\frac{1}{5\,!}+\cdots+\frac{1}{k\,!}."$$

☐ True. ☐ False. (Check one.)

(d1) Give one definition of e.

$$e \;=\; \lim_{n \to \infty}\left(1+\boxed{}\right)^n .$$

(d2) Give another definition of e.

$$e = \lim_{n \to \infty}\left(1+\frac{1}{\boxed{}}+\frac{1}{\boxed{}}+\frac{1}{\boxed{}}+\frac{1}{\boxed{}}+\cdots+\frac{1}{\boxed{}}\right).$$

(d3) The decimal expression of e up to the first six place under the decimal point

$$\boxed{}\,.\,\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}\;\cdots\;.$$

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[III] (Take-home; 20pts) Prove that $\sqrt{3}$ is an irrational number.

Proof. Proof by contradiction. Suppose $\sqrt{3}$ is written as

$$\sqrt{3} = \frac{k}{m}$$

using some integers k and m $\big($where $m \neq 0\big)$.

First, if both k and m are divisible by $\boxed{3}$, then we may simultaneously divide both the numerator and the denominator by $\boxed{3}$ $\big($and the value of the fraction stays the same$\big)$. After that procedure, suppose both the numerator and the denominator still remain to be divisible by $\boxed{}$, then we repeat the same procedure as many times as necessary until at least one of the numerator and the denominator is not divisible by $\boxed{}$. Thus we may assume, without loss of generality, that at least one of k and m is ____________________ .

Under this assumption, square the both sides of the identity $\sqrt{3} = \frac{k}{m}$, thus

$$3 = \frac{\boxed{}}{\boxed{}}.$$

This is the same as

$$\boxed{} = k^2.$$

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([III] continued)

The left-hand side of this last identity is clearly divisible by $\square$, so this last identity forces its right-hand side to be ________________________ .

That in turn implies k is ________________________ , because if k is ________________________ , then k^2 is ________________________ .

But then k being divisible by $\square$ implies k^2 is ____________________ .

So by virtue of the above last identity $3\,m^2$ is divisible by $\square$ or the same to say, m^2 is divisible by $\square$. This implies that m is ____________________ .

In short, both k and m are ____________________ .

This contradicts our assumption. The proof is complete. $\square$