

Math 105 TOPICS IN MATHEMATICS

REVIEW OF LECTURES – XIV

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§14. n -TH ROOT. FRACTIONAL EXPONENTS.

- After square-rooting, nothing stops us from considering “cube-rooting”:

$\sqrt[3]{0}$:	“ cube root of zero. ”
$\sqrt[3]{1}$:	“ cube root of one. ”
$\sqrt[3]{2}$:	“ cube root of two. ”
$\sqrt[3]{3}$:	“ cube root of three. ”
$\sqrt[3]{4}$:	“ cube root of four. ”
$\sqrt[3]{5}$:	“ cube root of five. ”
$\sqrt[3]{6}$:	“ cube root of six. ”
$\sqrt[3]{7}$:	“ cube root of seven. ”
$\sqrt[3]{8}$:	“ cube root of eight. ”

Again, explaining these is not too straightforward. Below is what you saw in “Review of Lectures – VI”:

x	0	1	2	3	4	5	6	7	8	9
x^3	0	1	8	27	64	125	216	343	512	729

Now, these mean

$$\begin{aligned}0^3 &= 0, \\1^3 &= 1, \\2^3 &= 8, \\3^3 &= 27, \\4^3 &= 64, \\5^3 &= 125, \\6^3 &= 216, \\7^3 &= 343, \\8^3 &= 512, \quad \text{and} \\9^3 &= 729,\end{aligned}$$

These translate into

$$\begin{aligned}\sqrt[3]{0} &= 0, \\ \sqrt[3]{1} &= 1, \\ \sqrt[3]{8} &= 2, \\ \sqrt[3]{27} &= 3, \\ \sqrt[3]{64} &= 4, \\ \sqrt[3]{125} &= 5, \\ \sqrt[3]{216} &= 6, \\ \sqrt[3]{343} &= 7, \\ \sqrt[3]{512} &= 8, \quad \text{and} \\ \sqrt[3]{729} &= 9.\end{aligned}$$

Exercise 1. $\sqrt[3]{1000} = ?$ $\sqrt[3]{1331} = ?$ $\sqrt[3]{1728} = ?$ $\sqrt[3]{4913} = ?$

Consult the table below, if necessary.

x	10	11	12	13	14	15	16	17	18	19
x^3	1000	1331	1728	2197	2744	3375	4096	4913	5832	6859

Answers	$\sqrt[3]{1000} = 10.$	Indeed,	$1000 = 10^3.$
	$\sqrt[3]{1331} = 11.$	Indeed,	$1331 = 11^3.$
	$\sqrt[3]{1728} = 12.$	Indeed,	$1728 = 12^3.$
	$\sqrt[3]{4913} = 17.$	Indeed,	$4913 = 17^3.$

So, in short:

If n is a non-negative integer, and if $a = n^3$ then $\sqrt[3]{a} = n$.

So, basically, the cube-rooting is the “inverse operation” to the cubing operation. Namely, suppose you cube a number. Then cube-root that outcome number. The effect is you “undo” the original cubing. What I mean is:

you cube 2	\implies	answer is 8	\implies	you cube-root 8	\implies	answer is 2.
you cube 3	\implies	answer is 27	\implies	you cube-root 27	\implies	answer is 3.
you cube 4	\implies	answer is 64	\implies	you cube-root 64	\implies	answer is 4.
you cube 5	\implies	answer is 125	\implies	you cube-root 125	\implies	answer is 5.
\vdots		\vdots		\vdots		\vdots

But the real issue here is,

$\sqrt[3]{2} = ?$	$\sqrt[3]{3} = ?$	$\sqrt[3]{4} = ?$	$\sqrt[3]{5} = ?$	$\sqrt[3]{6} = ?$	$\sqrt[3]{7} = ?$
$\sqrt[3]{9} = ?$	$\sqrt[3]{10} = ?$	$\sqrt[3]{11} = ?$	$\sqrt[3]{12} = ?$	$\sqrt[3]{13} = ?$	$\sqrt[3]{14} = ?$
$\sqrt[3]{15} = ?$	$\sqrt[3]{16} = ?$	$\sqrt[3]{17} = ?$	$\sqrt[3]{18} = ?$	$\sqrt[3]{19} = ?$	$\sqrt[3]{20} = ?$
$\sqrt[3]{21} = ?$	$\sqrt[3]{22} = ?$	$\sqrt[3]{23} = ?$	$\sqrt[3]{24} = ?$	$\sqrt[3]{25} = ?$	$\sqrt[3]{26} = ?$

etc. (as you can see, I excluded $\sqrt[3]{0}$, $\sqrt[3]{1}$, $\sqrt[3]{8}$ and $\sqrt[3]{27}$).

- $\sqrt[3]{2}$.

For example, $\sqrt[3]{2}$ means a number whose cube equals 2. Namely:

$$\text{“ } \boxed{x = \sqrt[3]{2}} \text{ is a number satisfying } \boxed{x^3 = 2} \text{ ”}$$

Here, we ask the same question as last time: “Does such a number exist?” The answer is, yes, such a number indeed exists. This is just like last time we asserted that $\sqrt{2}$ exists. How do we find $\sqrt[3]{2}$? The procedure is entirely parallel to how we found $\sqrt{2}$. It goes step by step.

0. Observe

$$\begin{aligned} 1^3 &= 1, && \longleftarrow \text{smaller than } 2 \\ 2^3 &= 8. && \longleftarrow \text{bigger than } 2 \end{aligned}$$

So $\sqrt[3]{2}$ must sit between 1 and 2:

$$1 < \sqrt[3]{2} < 2.$$

1. Observe

$$\begin{aligned} 1.1^3 &= 1.331, \\ 1.2^3 &= 1.728, && \longleftarrow \text{smaller than } 2 \\ 1.3^3 &= 2.197, && \longleftarrow \text{bigger than } 2 \end{aligned}$$

So $\sqrt[3]{2}$ must sit between 1.2 and 1.3:

$$1.2 < \sqrt[3]{2} < 1.3.$$

2. Observe

$$\begin{aligned} 1.21^3 &= 1.771561, \\ 1.22^3 &= 1.815848, \\ 1.23^3 &= 1.860867, \\ 1.24^3 &= 1.906624, \\ 1.25^3 &= 1.953125, & \longleftarrow \text{smaller than } 2 \\ 1.26^3 &= 2.000376. & \longleftarrow \text{bigger than } 2 \end{aligned}$$

So $\sqrt[3]{2}$ must sit between 1.25 and 1.26:

$$1.25 < \sqrt[3]{2} < 1.26.$$

3. Observe

$$\begin{aligned} 1.251^3 &= 1.957816251, \\ 1.252^3 &= 1.962515008, \\ 1.253^3 &= 1.967221277, \\ 1.254^3 &= 1.971935064, \\ 1.255^3 &= 1.976656375, \\ 1.256^3 &= 1.981385216, \\ 1.257^3 &= 1.986121593, \\ 1.258^3 &= 1.990865512, \\ 1.259^3 &= 1.995616979, & \longleftarrow \text{smaller than } 2 \\ 1.260^3 &= 2.000376000. & \longleftarrow \text{bigger than } 2 \end{aligned}$$

So $\sqrt[3]{2}$ must sit between 1.259 and 1.260:

$$1.259 < \sqrt[3]{2} < 1.260.$$

4. Observe

$$\begin{aligned} 1.2591^3 &= 1.996092541071, \\ 1.2592^3 &= 1.996568178688, \\ 1.2593^3 &= 1.997043891857, \\ 1.2594^3 &= 1.997519680584, \\ 1.2595^3 &= 1.997995544875, \\ 1.2596^3 &= 1.998471484736, \\ 1.2597^3 &= 1.998947500173, \\ 1.2598^3 &= 1.999423591192, \\ 1.2599^3 &= 1.999899757799, & \longleftarrow \text{smaller than } 2 \\ 1.2600^3 &= 2.000376000000. & \longleftarrow \text{bigger than } 2 \end{aligned}$$

So $\sqrt[3]{2}$ must sit between 1.2599 and 1.2600:

$$1.2599 < \sqrt[3]{2} < 1.2600.$$

5. Observe

$$\begin{aligned} 1.25991^3 &= 1.999947378617271, \\ 1.25992^3 &= 1.999995000191488, & \longleftarrow \text{smaller than } 2 \\ 1.25993^3 &= 2.000042622521657. & \longleftarrow \text{bigger than } 2 \end{aligned}$$

So $\sqrt[3]{2}$ must sit between 1.25992 and 1.25993:

$$1.25992 < \sqrt[3]{2} < 1.25993.$$

6. Observe

$$1.259921^3 = 1.999999762390486961, \quad \longleftarrow \text{smaller than } 2$$

$$1.259922^3 = 2.000004524597045448. \quad \longleftarrow \text{bigger than } 2$$

So $\sqrt[3]{2}$ must sit between 1.259921 and 1.259922:

$$1.259921 < \sqrt[3]{2} < 1.259922.$$

So,

$$\sqrt[3]{2} = 1.259921\dots$$

But of course, the above is only up to the sixth digit under decimal point of $\sqrt[3]{2}$. In other words, $\sqrt[3]{2}$ does not exactly equal 1.259921. But you can determine the seventh, eighth, ninth, ... digits, by simply continuing the above procedure. Indeed, the above procedure continues endlessly. If you want to see more digits:

$$\sqrt[3]{2} = 1.2599210498948731647672106072782283505702514647015\dots$$

Most importantly, the decimal expression of $\sqrt[3]{2}$ continues forever, it never ends.

- There is an algorithm for cube-rooting similar to the one for square-rooting which we have practiced last time. It determines the digit of the decimal expression of the cube-root of any given number at any place under the decimal point. We will not cover it. Something worth pointing out: What's underlying that method is, once again, Binomial Formula. Probably the same method is what our computers employ in calculating the cube-roots (or some minor variations of it).

- **Example 1.**

$$\sqrt[3]{3} = 1.44224957030740838232163831078010958839186925349935\dots,$$

$$\sqrt[3]{4} = 1.58740105196819947475170563927230826039149332789985\dots,$$

$$\sqrt[3]{5} = 1.70997594667669698935310887254386010986805511054305\dots,$$

$$\sqrt[3]{6} = 1.81712059283213965889121175632726050242821046314121\dots,$$

$$\sqrt[3]{7} = 1.91293118277238910119911683954876028286243905034587\dots$$

For the rest of today's class I want to focus on the theoretical aspect of the square-rooting, cube-rooting, and their generalizations. First, nothing stops us from considering the fourth-root, fifth-root, and so on so forth:

$$\sqrt[4]{2} \quad : \quad \text{“ fourth root of two. ”}$$

$$\sqrt[5]{3} \quad : \quad \text{“ fifth root of three. ”}$$

etc. The meaning:

$$\text{“ } \boxed{x = \sqrt[4]{2}} \text{ } \underline{\text{is a number satisfying}} \text{ } \boxed{x^4 = 2} \text{ ”}$$

$$\text{“ } \boxed{x = \sqrt[5]{3}} \text{ } \underline{\text{is a number satisfying}} \text{ } \boxed{x^5 = 3} \text{ ”}$$

More generally, for an arbitrary positive integer n , we may define the n -th root of a number.

Definition (n -th root).

Assume a is a positive number: $a > 0$. (Here, we do not assume that a is an integer. For example, a can be e .) Also, let n be a positive integer. Then

$$\text{“ } \boxed{x = \sqrt[n]{a}} \text{ } \underline{\text{is a number satisfying}} \text{ } \boxed{x^n = a} \text{ ”}$$

We call $\sqrt[n]{a}$ the n -th root of a .

Now, there is a term that refers to all of the square-root, the cube-root, the fourth-root, the fifth-root, and so on. They are called

“radicals.”

Also, the symbol ${}^n\sqrt{\quad}$ is called the radical symbol, or just the radical:

$${}^n\sqrt{\quad} : \text{ radical symbol } (\text{radical}).$$

★ So, for $n = 2$, ${}^n\sqrt{a}$ is ${}^2\sqrt{a}$, and this is just the square-root of a . There is absolutely nothing wrong in writing the square-root of a as ${}^2\sqrt{a}$, but it is customary that we allow ourselves to omit the tiny 2 in front of the radical symbol. So we usually write \sqrt{a} for ${}^2\sqrt{a}$.

• Below we consider three different situations where more than one radicals are involved in the same formation, or differen formations involving radicals can be identified:

• A. Question: Can we simplify

$$\sqrt{\sqrt{2}} \quad ?$$

Likewise, can we simplify

$$\sqrt[3]{\sqrt{2}} \quad ?$$

Or, can we simplify

$$\sqrt[4]{\sqrt[3]{2}} \quad ?$$

These are actually simplified as ${}^4\sqrt{2}$, ${}^6\sqrt{2}$, and ${}^{12}\sqrt{2}$, respectively. To highlight:

$$\sqrt{\sqrt{2}} = {}^4\sqrt{2},$$

$$\sqrt[3]{\sqrt{2}} = {}^6\sqrt{2},$$

$$\sqrt[4]{\sqrt[3]{2}} = {}^{12}\sqrt{2}.$$

In the above what's inside the inner radical is 2. A more general formula is below:

Rule A.

$$\boxed{\sqrt[n]{\sqrt[k]{a}} = \sqrt[nk]{a}} .$$

- **B.** Rule A has the following applications:

Example 2. $\sqrt[4]{4}$ is simplified as $\sqrt{2}$. Indeed:

$$\sqrt[4]{4} = \sqrt{\sqrt{4}} = \sqrt{2} .$$

Similarly:

Example 3. $\sqrt[6]{25}$ is simplified as $\sqrt[3]{5}$. Indeed:

$$\sqrt[6]{25} = \sqrt[3]{\sqrt{25}} = \sqrt[3]{5} .$$

Example 4. $\sqrt[15]{27}$ is simplified as $\sqrt[5]{3}$. Indeed:

$$\sqrt[15]{27} = \sqrt[5]{\sqrt[3]{27}} = \sqrt[5]{3} .$$

More generally:

$$2k\sqrt{a^2} = k\sqrt{a} ,$$

$$3k\sqrt{a^3} = k\sqrt{a} ,$$

$$4k\sqrt{a^4} = k\sqrt{a} ,$$

$$5k\sqrt{a^5} = k\sqrt{a} ,$$

⋮

To highlight:

Rule B.

$$\boxed{{}^{nk}\sqrt{a^n} = {}^k\sqrt{a}} .$$

Exercise 2. Use Rule A to simplify

$$(1) \quad {}^3\sqrt{\sqrt{6}}. \quad (2) \quad {}^4\sqrt{{}^3\sqrt{10}}.$$

$$\left[\underline{\text{Answers}} \right]: \quad (1) \quad {}^6\sqrt{6}. \quad (2) \quad {}^{12}\sqrt{10}.$$

Exercise 3. Use Rule B to simplify

$$(1) \quad {}^6\sqrt{125}. \quad (2) \quad {}^{12}\sqrt{343}. \quad (3) \quad {}^{16}\sqrt{256}.$$

$$(4) \quad {}^{12}\sqrt{81}.$$

$$\left[\underline{\text{Answers}} \right]: \quad (1) \quad \sqrt{5}. \quad (2) \quad {}^4\sqrt{7}. \quad (3) \quad \sqrt{2}.$$

$$(4) \quad {}^3\sqrt{3}.$$

• C. Question: Are

$$\sqrt{2} \cdot \sqrt{3} \quad \text{and} \quad \sqrt{6}$$

the same, in view of the fact that $2 \cdot 3 = 6$?

Likewise, are

$${}^3\sqrt{5} \cdot {}^3\sqrt{7} \quad \text{and} \quad {}^3\sqrt{35}$$

the same, in view of the fact that $5 \cdot 7 = 35$?

The answer is, ‘yes indeed’. So,

$$\begin{aligned}\sqrt{2} \cdot \sqrt{3} &= \sqrt{6}, \\ \sqrt[3]{5} \cdot \sqrt[3]{7} &= \sqrt[3]{35}.\end{aligned}$$

More generally:

$$\begin{aligned}(\sqrt[2]{a})(\sqrt[2]{b}) &= \sqrt[2]{ab}, \\ (\sqrt[3]{a})(\sqrt[3]{b}) &= \sqrt[3]{ab}, \\ (\sqrt[4]{a})(\sqrt[4]{b}) &= \sqrt[4]{ab}, \\ (\sqrt[5]{a})(\sqrt[5]{b}) &= \sqrt[5]{ab}, \\ (\sqrt[6]{a})(\sqrt[6]{b}) &= \sqrt[6]{ab}, \\ &\vdots\end{aligned}$$

To highlight:

Rule C.

$$\boxed{(\sqrt[n]{a})(\sqrt[n]{b}) = \sqrt[n]{ab}}.$$

This can be immediately generalized as follows:

Rule C'.

$$\boxed{(\sqrt[n]{a})(\sqrt[n]{b})(\sqrt[n]{c}) = \sqrt[n]{abc}}.$$

Rule C''.

$$\boxed{(\sqrt[n]{a})(\sqrt[n]{b})(\sqrt[n]{c})(\sqrt[n]{d}) = \sqrt[n]{abcd}}.$$

Exercise 4. Use Rule C (or Rule C', or Rule C'') to simplify

$$(1) \quad \sqrt{6} \cdot \sqrt{11}. \quad (2) \quad \sqrt[3]{3} \cdot \sqrt[3]{5}. \quad (3) \quad \sqrt[4]{2} \cdot \sqrt[4]{5} \cdot \sqrt[4]{13}.$$

$$(4) \quad \sqrt[5]{2} \cdot \sqrt[5]{3} \cdot \sqrt[5]{4} \cdot \sqrt[5]{5}.$$

$$\boxed{\text{Answers}}: \quad (1) \quad \sqrt{66}. \quad (2) \quad \sqrt[3]{15}. \quad (3) \quad \sqrt[4]{130}. \quad (4) \quad \sqrt[5]{120}.$$

• **D.** Rule C has the following applications (the case $a = b$):

$$\begin{aligned} (\sqrt{a})^2 &= \sqrt{a^2} = a, \\ (\sqrt[3]{a})^2 &= \sqrt[3]{a^2}, \\ (\sqrt[4]{a})^2 &= \sqrt[4]{a^2} = \sqrt{a}, \\ (\sqrt[5]{a})^2 &= \sqrt[5]{a^2}, \\ (\sqrt[6]{a})^2 &= \sqrt[6]{a^2} = \sqrt[3]{a}, \\ &\vdots \end{aligned}$$

Similarly, Rule C' has the following applications (the case $a = b = c$):

$$\begin{aligned} (\sqrt{a})^3 &= \sqrt{a^3}, \\ (\sqrt[3]{a})^3 &= \sqrt[3]{a^3} = a, \\ (\sqrt[4]{a})^3 &= \sqrt[4]{a^3}, \\ (\sqrt[5]{a})^3 &= \sqrt[5]{a^3}, \\ (\sqrt[6]{a})^3 &= \sqrt[6]{a^3} = \sqrt{a}, \\ &\vdots \end{aligned}$$

Similarly, Rule C'' has the following applications (the case $a = b = c = d$):

$$\begin{aligned}(\sqrt{a})^4 &= \sqrt{a^4}, \\(3\sqrt{a})^4 &= 3\sqrt{a^4}, \\(4\sqrt{a})^4 &= 4\sqrt{a^4} = a, \\(5\sqrt{a})^4 &= 5\sqrt{a^4}, \\(6\sqrt{a})^4 &= 6\sqrt{a^4} = 3\sqrt{a^2}, \\(7\sqrt{a})^4 &= 7\sqrt{a^4}, \\(8\sqrt{a})^4 &= 8\sqrt{a^4} = \sqrt{a}, \\&\vdots\end{aligned}$$

To highlight:

Rule D.

$$\boxed{\left({}^k\sqrt{a}\right)^n = {}^k\sqrt{a^n}}.$$

★ Now, you might at first feel the above rules are slightly confusing. There is a different notation that helps understand these.

New notation.

$$\boxed{\sqrt[n]{a} = a^{\frac{1}{n}}}.$$

This is the first time we see fractional powers. The virtue of this way of writing is as follows: The above rules (Rule A, Rule B, Rule C, Rule C', Rule C'', and Rule D) can be rewritten as follows:

Rule A Paraphrased.
$$\left(a^{\frac{1}{k}}\right)^{\frac{1}{n}} = a^{\frac{1}{nk}}.$$

Rule B Paraphrased.
$$\left(a^n\right)^{\frac{1}{nk}} = a^{\frac{1}{k}}.$$

Rule C Paraphrased.
$$\left(a^{\frac{1}{n}}\right)\left(b^{\frac{1}{n}}\right) = \left(ab\right)^{\frac{1}{n}}.$$

Rule C' Paraphrased.
$$\left(a^{\frac{1}{n}}\right)\left(b^{\frac{1}{n}}\right)\left(c^{\frac{1}{n}}\right) = \left(abc\right)^{\frac{1}{n}}.$$

Rule C'' Paraphrased.
$$\left(a^{\frac{1}{n}}\right)\left(b^{\frac{1}{n}}\right)\left(c^{\frac{1}{n}}\right)\left(d^{\frac{1}{n}}\right) = \left(abcd\right)^{\frac{1}{n}}.$$

Rule D Paraphrased.
$$\left(a^{\frac{1}{k}}\right)^n = \left(a^n\right)^{\frac{1}{k}}.$$

Once you see the same rules written in this format, their meaning becomes more transparent. — To be continued.