Math 105 TOPICS IN MATHEMATICS

REVIEW OF LECTURES – XV

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§15. Fractional exponents.

• Right now (until "Review of Lectures – XVII") I am shooting for defining

$$2^x, \qquad 3^x, \qquad 4^x, \quad \cdots$$

as functions dependent on x. These are called <u>exponential functions</u> . x on the right shoulder of 2, 3, 4, $\cdots$ is called <u>the exponent</u> . Today we cover some preliminary materials. Last time I briefly touched fractional exponents. Today we are going to push this direction further. Let's start with some review. First, we have defined "the n-th root" as a generalization of the square-root:

Definition (n-th root).

Assume a is a positive number: $a > 0$. (Here, we do not assume that a is an integer. For example, a can be e.) Also, let n be a positive integer. Then

$$\text{“} \boxed{x = \sqrt[n]{a}} \quad \underline{\underline{\textit{is a number satisfying}}} \quad \boxed{x^n = a} \text{”}.$$

We call $\sqrt[n]{a}$ <u>the n-th root of a .</u>

So, for $n = 2$ this is the squre-root, for $n = 3$ this is the cube-root, for $n = 4$ this is the fourth-root, and so on. There is a term that refers to these operations, that is:

"radicals."

⋆ Also, as for the name of the symbol:

$$\sqrt[n]{} \quad : \quad \text{radical symbol} \quad (\text{radical}).$$

We have learned some rules about various formations involving radicals. Let me highlight those.

- Below n and k are integers, and a is a positive number (not necessarily an integer).

Rule A. $$\boxed{\sqrt[n]{\sqrt[k]{a}} \quad = \quad \sqrt[nk]{a}}\ .$$

Rule B. $$\boxed{\sqrt[nk]{a^n} \quad = \quad \sqrt[k]{a}}\ .$$

Rule C. $$\boxed{\left(\sqrt[n]{a}\right)\left(\sqrt[n]{b}\right) \quad = \quad \sqrt[n]{ab}}\ .$$

Rule C′. $$\boxed{\left(\sqrt[n]{a}\right)\left(\sqrt[n]{b}\right)\left(\sqrt[n]{c}\right) \quad = \quad \sqrt[n]{abc}}\ .$$

Rule C″. $$\boxed{\left(\sqrt[n]{a}\right)\left(\sqrt[n]{b}\right)\left(\sqrt[n]{c}\right)\left(\sqrt[n]{d}\right) \quad = \quad \sqrt[n]{abcd}}\ .$$

Rule D. $$\boxed{\left(\sqrt[k]{a}\right)^n \quad = \quad \sqrt[k]{a^n}}\ .$$

Then we have introduced an alternative notation for the radicals:

- **Alternative Notation.**

$$\boxed{\sqrt[n]{a} \quad = \quad a^{\frac{1}{n}}}\ .$$

We have rewritten the above set of rules, using this new notation, as follows:

Rule A Paraphrased. $$\left(a^{\frac{1}{k}}\right)^{\frac{1}{n}} \quad = \quad a^{\frac{1}{nk}}.$$

Rule B Paraphrased. $$\left(a^n\right)^{\frac{1}{nk}} \quad = \quad a^{\frac{1}{k}}.$$

Rule C Paraphrased. $$\left(a^{\frac{1}{n}}\right)\left(b^{\frac{1}{n}}\right) \quad = \quad \left(ab\right)^{\frac{1}{n}}.$$

Rule C′ Paraphrased. $$\left(a^{\frac{1}{n}}\right)\left(b^{\frac{1}{n}}\right)\left(c^{\frac{1}{n}}\right) \quad = \quad \left(abc\right)^{\frac{1}{n}}.$$

Rule C″ Paraphrased. $$\left(a^{\frac{1}{n}}\right)\left(b^{\frac{1}{n}}\right)\left(c^{\frac{1}{n}}\right)\left(d^{\frac{1}{n}}\right) \quad = \quad \left(abcd\right)^{\frac{1}{n}}.$$

Rule D Paraphrased. $$\left(a^{\frac{1}{k}}\right)^{n} \quad = \quad \left(a^n\right)^{\frac{1}{k}}.$$

And that was the review.

- Today I want to start with the following important definition:

Definition. $\underline{\underline{\text{Define}}}$ $\quad a^{\frac{n}{k}} \quad$ $\underline{\underline{\text{as}}}$ $\quad \left(a^n\right)^{\frac{1}{k}}.$

By virtue of 'Rule D Paraphrased' above, we can give an equivalent definition:

Definition. $\underline{\underline{\text{Define}}}$ $\quad a^{\frac{n}{k}} \quad$ $\underline{\underline{\text{as}}}$ $\quad \left(a^{\frac{1}{k}}\right)^{n}.$

- So, what does this do? Yes, we have just defined

$$\boxed{\;a^r\;} \quad \underline{\underline{\text{where}}} \quad \boxed{\;r\;} \quad \underline{\underline{\text{is a rational number.}}}$$

You don't have to think that this is too abstract. The virtue of this definition is that we can now encaplsulate the above miscellaneous rules in a concise form:

Rule I. $$\boxed{\Big(ab\Big)^r \;=\; a^r\, b^r} \;.$$

Rule II. $$\boxed{a^r\; a^s \;=\; a^{r+s}} \;.$$

Rule III. $$\boxed{\Big(a^r\Big)^s \;=\; a^{rs}} \;.$$

Also

Rule IV. $$\boxed{a^0 \;=\; 1} \;, \qquad \boxed{1^r \;=\; 1} \;.$$

$\star$ Two remarks are in order. First, I could have supplied the following variations:

Rule I′. $$\Big(abc\Big)^r \;=\; a^r\, b^r\, c^r,$$

Rule I″. $$\Big(abcd\Big)^r \;=\; a^r\, b^r\, c^r\, d^r,$$

etc., just like Rule C′ and Rule C″ are accompanied with Rule C. Mathematically speaking, this is redundant. Indeed, using Rule I twice would yield Rule I′:

$$\begin{aligned}
\Big(abc\Big)^r &= \Big(ab\Big)^r\, c^r && \Big(\text{by Rule I}\Big) \\
&= \Big(a^r\, b^r\Big)\, c^r && \Big(\text{by Rule I}\Big) \\
&= a^r\, b^r\, c^r.
\end{aligned}$$

Similarly, using Rule I three times would yield Rule I″. (Now, due to the same reason, in retrospect, Rule C′ and Rule C″ were also redundant.)

Another remark: Rule II does not seem to follow from the rules in the previous lecture. But the reason why Rule II is valid can be seen as follows:

First, an easy special case:

$$\begin{aligned} a^2 \cdot a^3 &= a\ a\ a\ a\ a \\ &= a^5\ , \end{aligned}$$

$$\begin{aligned} a^3 \cdot a^4 &= a\ a\ a\ a\ a\ a\ a \\ &= a^7\ , \end{aligned}$$

and so on. You can extrapolate and conclude

$$a^n\ a^\ell \quad = \quad a^{n+\ell} \qquad\qquad \Big(n,\ \ \ell\ :\ \ \text{positive } \underline{\underline{\text{integers}}}\ \Big).$$

Now, Rule II asserts

$$a^r\ a^s \quad = \quad a^{r+s} \qquad\qquad \Big(r,\ \ s\ :\ \ \text{positive } \underline{\underline{\text{rational numbers}}}\ \Big),$$

which is more general than the above. To see that the latter is indeed true, we first test it by some concrete example. Suppose

$$r\ =\ \frac{1}{4} \qquad \text{and} \qquad s\ =\ \frac{5}{6}.$$

then

$$r\ =\ \frac{1\ \cdot 3}{4\ \cdot 3}\ =\ \frac{3}{12} \qquad \text{and} \qquad r\ =\ \frac{5\ \cdot 2}{6\ \cdot 2}\ =\ \frac{10}{12}.$$

$\big($common denominator technique, see "Supplement"$\big)$. Accordingly,

$$
\begin{aligned}
r + s &= \frac{3}{12} + \frac{10}{12} \\
&= \frac{3 + 10}{12} \\
&= \frac{13}{12}.
\end{aligned}
$$

Meanwhile,

$$
\begin{aligned}
a^r\ a^s &= a^{\frac{3}{12}}\ a^{\frac{10}{12}} \\
&= \left(a^{\frac{1}{12}}\right)^3 \left(a^{\frac{1}{12}}\right)^{10}.
\end{aligned}
$$

At this point we treat $a^{\frac{1}{12}}$ as an individual number, call it b, so the above is

$$b^3\ b^{10} = b^{13}.$$

Remember that $b = a^{\frac{1}{12}}$, so this last outcome b^{13} becomes

$$\left(a^{\frac{1}{12}}\right)^{13}.$$

By definition, this equals

$$a^{\frac{13}{12}}.$$

To conclude, $a^r\ a^s = a^{r+s}$ is indeed true for $r = \dfrac{1}{4}$ and $s = \dfrac{5}{6}$.

It is now easy to take care of the case r and s are arbitrary positive rational numbers, by generalizing the above argument. Namely, write r and s as

$$r = \frac{n}{k}, \quad s = \frac{\ell}{k},$$

using appropriate positive integers n, ℓ and k, which is always feasible $\big($common denominator technique, see "Supplement"$\big)$. Then

$$\begin{aligned} a^r\, a^s &= a^{\frac{n}{k}}\, a^{\frac{\ell}{k}} \\ &= \left(a^{\frac{1}{k}}\right)^n \left(a^{\frac{1}{k}}\right)^\ell. \end{aligned}$$

At this point we may treat $a^{\frac{1}{k}}$ as an individual number, call it b, so the above is

$$b^n\, b^\ell.$$

Here n and ℓ are positive integers, thus we previously saw that this equals $b^{n+\ell}$. Now, remember that b equals $a^{\frac{1}{k}}$, so

$$\begin{aligned} b^{n+\ell} &= \left(a^{\frac{1}{k}}\right)^{n+\ell} \\ &= a^{\frac{n+\ell}{k}} \\ &= a^{\frac{n}{k}+\frac{\ell}{k}}. \end{aligned}$$

This is nothing else but a^{r+s}. In short, $a^r\, a^s = a^{r+s}$. This establishes Rule II.

$\star$ Now, we may also highlight some variations of Rule II, such as

$$a^r\ a^s\ a^t \quad = \quad a^{r+s+t},$$

$$a^r\ a^s\ a^t\ a^u \quad = \quad a^{r+s+t+u},$$

etc., but these are redundant, just the same reason Rule I′, Rule I″, *etc.* are redundant. Indeed, these are immediate consequences of Rule II (apply Rule II repeatedly).

The above rules (Rules I–IV) are usually put together, and are referred to as the exponential laws. So let me highlight them one more time:

Exponential Laws. Let r and s be positive rational numbers. Let a and b be positive numbers (not necessarily rational numbers). Then

Rule I. $$\boxed{\left(ab\right)^r \quad = \quad a^r\, b^r}\ .$$

Rule II. $$\boxed{a^r\ a^s \quad = \quad a^{r+s}}\ .$$

Rule III. $$\boxed{\left(a^r\right)^s \quad = \quad a^{rs}}\ .$$

Rule IV. $$\boxed{a^0 \ = \ 1}\ , \qquad \boxed{1^r \ = \ 1}\ .$$

Exercise 1. Simplify

(1) $2^{\frac{2}{3}} \cdot \left(\frac{3}{2}\right)^{\frac{2}{3}}$. (2) $3^{\frac{1}{2}} \cdot 3^{\frac{5}{2}}$. (3) $\left(2^3\right)^{\frac{1}{4}}$.

(4) 100^0. (5) $1^{\frac{4}{7}}$.

[**Answers**]: (1) $3^{\frac{2}{3}}$. (2) $3^3 = 27$. (3) $2^{\frac{3}{4}}$.

(4) 1. (5) 1.

- **Continuity.**

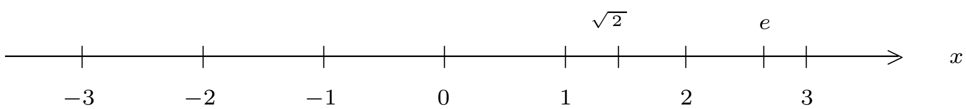
One of the important aspects of fractional exponents, which is not clear at all from the definition given above, is the following:

“ *When r and s are close, then a^r and a^s are close.* ”

This may sound innocuous. A savvy reader will immediately detect that the level of precision in this phrase is unsatisfactory in the modern standards of rigor in mathematics. I am actually going to tell you that there is a way to make this sentence precise (see page 19). It is ‘crafty’, though. Along the way I am going to explain why that is important. First I should better give some background information (a quick run-down):

- **What are numbers, after all?**

For a long time in the history of math, the concept of numbers used to be taken for granted. Mathematicians used to simply resort to the straightline as a mathematical model for numbers, and it sufficed.



The idea is

“ *line is continuous, it is nowhere broken.* ”

Then at some point they started encountering mathematical phenomena whose complete breakdown hinged on a deeper grasp of the concept of numbers. Mathematicians suddenly retrospectively realize that their understanding of ‘numbers’ rooted in such postulation “line is nowhere broken” was inadequate for what they needed. After all, ‘line’ is an undefined term. By the 19th century it became increasingly clear that mathematicians needed to reach a consensus once and for all about the mathematially clear-cut, completely self-contained, definition of numbers, with no reliance on geometric intuitions such as ‘lines’. They asked themselves: “What are numbers after all?” Because without it, you cannot even answer

Question. “*What is the largest number*?”

The answer is, of course, "no, there is no such thing as the largest number". Yet technically, without a mathematically precise definition of numbers it would be impossible to logically refute the existence of the largest number. The same goes for

Question. "*Is* 1 and 0.9999999999999... (*the digit* 9 *repeats forever*) *the same, or different*?"

The answer is, "they are the same", but in order to justify it, once again, you would have to resort to the precise definition of numbers.

- **Real numbers, complex numbers.**

First, they had to give a name for exactly what objects they are dealing with. True, the objects are called "numbers". So what's wrong with it, just calling them as 'numbers', right? But as it transpired, mathematicians are clearly better off (i) recognizing all different classes of numbers, and (ii) clearly setting the outer boundary of each of those classes of numbers. What that means is, for example, we all know integers form one class of numbers, and rational numbers form another. It turned out this way of categorization of numbers is fruitful. Meanwhile, you have learned that irrational numbers are those that are not rational. But here you have to ask twice, "within which class of numbers"? This might at first sound absurd, or meaningless. But this is imperative when you deal with numbers like $\sqrt{-1}$.

Okay, that was just out of the blue. In fact, in this Math 105 class, so far we have thoroughly precluded something like $\sqrt{-1}$. Indeed, at the beginning of today's lecture, I clearly made an assumption that a is positive, and kept that assumption intact each time I raised some fractional powers on top of a. But $\sqrt{-1}$ naturally arises in the context of solving a certain type of equations that casually crop up in mathematics (quadratic equations), whose detailed discussion I will rather spare today. Making-a-long-story-short version of it is, people at first struggled to accept something like $\sqrt{-1}$, mainly for the psychological reason. But then soon the benefits of implementing $\sqrt{-1}$ would outweigh the psychological stigma, because with the aid of $\sqrt{-1}$ there are million things mathematicians can do (other than just solving quadratic equations) even within the contexts in which the presence of $\sqrt{-1}$ is impalpable. So, finally $\sqrt{-1}$ was enfranchised in mathematics. And that was in the late 18th century. Now, then suddenly there arose a need to 'profile' the 'usual' numbers, the numbers that do not involve $\sqrt{-1}$. Indeed, the development of math has reached the stage where its further progress — with effectively incorporating $\sqrt{-1}$ — was strictly dependent on the precise definition of the 'usual' numbers.

It was in that context that mathematicians came up with the names

$$\text{“real numbers,”} \qquad \text{“complex numbers.”}$$

Complex numbers form the broader class of the two. Complex numbers are something like $1 + 2\sqrt{-1}$, or $-3 + \sqrt{-2}$, where $\sqrt{-2}$ is $\sqrt{2} \cdot \sqrt{-1}$, *etc.* More generally, any complex number is written as

$$a \,+\, b\sqrt{-1} \qquad \Big(a, \;\; b \;:\; \;\; \text{real numbers}\Big).$$

When $b = 0$, the part $\sqrt{-1}$ disappears, so it is just a real number. It is precisely in this sense that complex numbers are generalizations of real numbers, or 'usual' numbers. "Informally", complex numbers are two-dimensional, in the sense each complex number has two real number components a and b, each of which is one-dimensional, as in each of a and b independently runs through a straightline (again, "informally"). By the way, those complex numbers which are not real are called 'imaginary numbers'. (Historically, the term 'imaginary number' came before the term 'complex number'.) The number $\sqrt{-1}$ is called the 'unit imaginary number'. Today, math without complex numbers — or math without $\sqrt{-1}$ — would be inconceivable. Complex numbers are absolutely an indispensable tool in every corner of math. In this class, I plan to touch this topic near the end of the semester.

• **Recap:** The very nature of the subject requires one to define the outer boundary of the 'usual' numbers. In that context, mathematicians have decided to call the 'usual' numbers as 'real numbers'. Then there is a broader class of numbers, called 'complex numbers'. Categorizing all different classes of numbers is paramount: (a) integers, (b) rational numbers, (c) real numbers, and (d) complex numbers. Here (b) is broader than (a); (c) is broader than (b), and (d) is broader than (c). In this Math 105 class, we have not talked about (d) yet, but we will.

• **Axiomatic definition of real numbers. Axiom of continuity.**

Now, let's forget about complex numbers. In what follows we will focus on real numbers. In the above paragraphs, I actually did not quite give you the precise definition of real numbers. Indeed, it is rather complicated. Mathematicians in the 19th century fought hard to come up with one. (Among others) the following realization served as the cornerstone: There are many diffferent levels of "infinitely many". What I am talking about is this: Everyone knows that there are infinitely many real numbers, and then there are also infinitely many rational numbers. But the former is a 'thicker', 'denser', kind of infinitely many. In other words, the way rational numbers are spread across the entire real numbers is very 'thin', or 'sparse'. And that gave people an ultimate clue. It didn't take long before they got hold of an undisputable, consensus definition of the real numbers. As it turns out, the best way to define real numbers is to resort it to *axioms*, and characterize real numbers by those *axioms*. Several equivalent versions of axioms that characterize real numbers are known, and mathematicians' names are attached to those axioms who are responsible for them (Cauchy, Weierstrass, Dedekind and Cantor, to name a few).* Each version is called

"the axiom of continuity."

The detailed discussion of the axiom of continuity belongs to a graduate Analysis course (Math 765 in our course listings) so I shall not offer any further insights about it, but a little more historical note:

- **Hilbert versus Gödel.**

Such re-examination of the concept of numbers would eventually intermingle with a much bigger movement to build the ultimate foundations on which the entire math should be logically built, and it would eventually be crystallized in the form of a brand-new math called 'foundation of math'. For example, Gödel's incompleteness theorem (see "Review of Lectures – XIII", page 21) belongs to 'foundation of math'.

Speaking of, I must mention the name Hilbert.** Hilbert actually came before Gödel (as in half century or so older), and was at the center of that movement. Hilbert would later become known as the grandfather of the 20th century math (which is 1000 times more complex and refined than the 19th century math), and the pretext of that is he gave a famous speech in 1900 in Paris, offering some blueprint of how math in the next 100 years should develop. Hilbert, as ingenious and profound a mathematician as he was, somehow didn't seem to believe that there can possibly exist a mathematical statement which is 'undecidable'. Then Gödel emerged and shocked Hilbert (and the world) by refuting Hilbert's belief by actually proving the incompleteness theorem (see "Review of Lectures – XIII", page 21). All this is something which all working mathematicians know. The truth be told, to most working mathematicians in pure (and applied) math (myself included), anything beyond that, and the axioms of continuity (and something called Zorn's lemma which you

*Augustin-Louis Cauchy (1789–1857), Karl Weierstrass (1815–1897), Richard Dedekind (1831–1916) and Georg Cantor (1845–1918).

**David Hilbert (1862–1943).

don't have to know) as far as 'foundation of math' goes is a rather unfamiliar territory of math (as in heart surgeons and brain surgeons possess different expertise). Mathematicians are generally 'pragmatic' within math, save they acknowledge the overwhelming psychological impact the incompleteness theorem gave to pure math.*

Now, for those of you who read this up until this point and say "why should we care". Let me tell you: Without the axiom of continuity, we cannot infer that, two circles whose centers are close enough relative to their radia but neither one surrounds another should intersect. What I mean is, there may be a break in the perimeters of those circles. The axiom of continuity guarantees us that such a break doesn't exist.

Now, all of a sudden let's return to our original problem: We were looking at the statement

" *When r and s are close, then a^r and a^s are close.* "

I said I was going to make this precise. This is actually classified as a problem of continuity, though the nuance is little bit different from all the above. Indeed, the latter addresses the continuity of 'the function' a^x, rather than the continuity of the real numbers themselves. So this is actually the next level. Yet, most importantly, without the solid underpinning structure of the continuity of real numbers, this latter inquiry is meaningless. Now, all the above background information gives you a better idea why we must address something like this. You are going to understand what it means that the way it is stated above is imprecise, and why the formulation that I am going to give you next fits the bill. Once again, the above inquiry of closeness lies at the heart of the subject of continuity.

A little sneak preview: I set my sights on defining the numbers like 2^x, 3^x, e^x, and so on, where x is a real number but not necessarily a rational number. The continuity is the key underlying reason why making such a definition is feasible. And 2^x, 3^x, e^x, *etc.* are so fundamental in mathematics as in those are on every precalculus and calculus textbook. Those textbooks (commercially circulated textbooks in US) may or maynot (most of them actually don't) provide the aforementioned background information pertaining to the continuity. So I am filling in the holes.

My goal today is to explain to you the idea of how the above statement of 'closeness' is made precise in the appropriate modern mathematics language, the way mathematicians today understand it. First we need to state some auxiliary fact. In mathematics, an auxiliary fact not quite worthy to call it a 'theorem' but needed to prove some other theorems is called a 'lemma'. So, a lemma:

*Pure mathematicians being 'pragmatic' sounds an 'oxymoron' since pure math is deemed as the most 'unpragmatic' among all scholarly disciplines, probably only to be paralleled with philosophy.

Lemma 1. Let r be a rational number between 0 and 1. Then

$$2^r \;<\; 1 + r.$$

Proof $\Big($ends at the end of page 16 — follow it only if you have a strong stomach$\Big)$**.**

Write r as $r = \dfrac{n}{k}$ using two positive integers n and k. We have $n < k$ by assumption. We need to prove

$$2^{\frac{n}{k}} \;<\; 1 + \frac{n}{k}.$$

In view of the fact that $a^k < b^k$ $\big(a$ and b are positive real numbers, and k is a positive integer$\big)$ implies $a < b$, it suffices to prove

$$\left(2^{\frac{n}{k}}\right)^k \;<\; \left(1 + \frac{n}{k}\right)^k.$$

Note that $\left(2^{\frac{n}{k}}\right)^k$ is simplified as 2^n, so it suffices to prove

$$2^n \;<\; \left(1 + \frac{n}{k}\right)^k.$$

Since $2 = 1 + 1$, so it suffices to prove

$$\Big(1 + 1\Big)^n \;<\; \left(1 + \frac{n}{k}\right)^k.$$

Our strategy of proof is binomially expand each of the two sides, and compare the respective terms. So, let's perform the binomial expansion of each of

$$\Big(1 + 1\Big)^n \qquad \Big(\text{the left-hand side}\Big), \quad \text{and}$$

$$\left(1 + \frac{n}{k}\right)^k \qquad \Big(\text{the right-hand side}\Big).$$

$$
\begin{aligned}
\text{the left-hand side} \quad = \quad & 1 && \text{(term L-0)} \\
& + \frac{n}{1\,!} && \text{(term L-1)} \\
& + \frac{n\left(n-1\right)}{2\,!} && \text{(term L-2)} \\
& + \frac{n\left(n-1\right)\left(n-2\right)}{3\,!} && \text{(term L-3)} \\
& + \frac{n\left(n-1\right)\left(n-2\right)\left(n-3\right)}{4\,!} && \text{(term L-4)} \\
& + \cdots \\
& + \frac{n\left(n-1\right)\left(n-2\right)\ \cdots\ 2 \cdot 1}{n\,!} && \text{(term L-}n\text{)}
\end{aligned}
$$

$$
\begin{aligned}
\text{the right-hand side} \quad = \quad & 1 && \text{(term R-0)} \\
& + \frac{k}{1\,!} \cdot \left(\frac{n}{k}\right) && \text{(term R-1)} \\
& + \frac{k\left(k-1\right)}{2\,!} \cdot \left(\frac{n}{k}\right)^2 && \text{(term R-2)} \\
& + \frac{k\left(k-1\right)\left(k-2\right)}{3\,!} \cdot \left(\frac{n}{k}\right)^3 && \text{(term R-3)} \\
& + \frac{k\left(k-1\right)\left(k-2\right)\left(k-3\right)}{4\,!} \cdot \left(\frac{n}{k}\right)^4 && \text{(term R-4)} \\
& + \cdots \\
& + \frac{k\left(k-1\right)\left(k-2\right)\ \cdots\ \left(k-n+1\right)}{n\,!} \cdot \left(\frac{n}{k}\right)^n && \text{(term R-}n\text{)} \\
& + \cdots
\end{aligned}
$$

Note that the left-hand side ends with (term L-n) whereas the right-hand side has terms beyond (term R-n). Agree that, regardless, all the terms on both sides are positive.

Now, in this situation, let's compare the respective terms. Since $\dfrac{n}{k} < 1$, we have

$$
\begin{aligned}
1 &= (\text{term L-0}) = (\text{term R-0}) = 1 \\
\frac{n}{1\,!} &= (\text{term L-1}) = (\text{term R-1}) = \frac{n}{1\,!} \\
\frac{n\left(n-1\right)}{2\,!} &= (\text{term L-2}) < (\text{term R-2}) = \frac{n\left(n-\frac{n}{k}\right)}{2\,!} \\
\frac{n\left(n-1\right)\left(n-2\right)}{3\,!} &= (\text{term L-3}) < (\text{term R-3}) = \frac{n\left(n-\frac{n}{k}\right)\left(n-\frac{2n}{k}\right)}{3\,!} \\
\frac{n\left(n-1\right)\left(n-2\right)\left(n-3\right)}{4\,!} &= (\text{term L-4}) < (\text{term R-4}) \\
&\qquad\qquad = \frac{n\left(n-\frac{n}{k}\right)\left(n-\frac{2n}{k}\right)\left(n-\frac{3n}{k}\right)}{4\,!} \\
&\quad\vdots \qquad\qquad \vdots \\
\frac{n\left(n-1\right)\left(n-2\right)\ \cdots\ 1}{n\,!} &= (\text{term L-}n) < (\text{term R-}n) \\
&= \frac{n\left(n-\frac{n}{k}\right)\left(n-\frac{2n}{k}\right)\ \cdots\ \left(n-\frac{\left(n-1\right)n}{k}\right)}{n\,!}.
\end{aligned}
$$

The above clearly shows that the right-hand side is bigger than the left-hand side. Our proof of Lemma 1 is complete. $\square$

- Finally, we are ready to address the following:

$$\text{“}\ \underline{\underline{\textit{When } r \textit{ and } s \textit{ are close, then } a^r \textit{ and } a^s \textit{ are close.}}}\ \text{”}$$

Below I am going to make this statement precise $\big($‘Theorem’ in page 19$\big)$. I want to be concrete, so at first I am going to throw a randomly chosen concrete number, say 3 $\big($it doesn’t have to be $3\big)$, into one of r and s. So, let’s say r. So $r = 3$. In addition, I am going to throw another number, say 2 $\big($it doesn’t have to be $2\big)$, into a. So $a = 2$. $\big($If you change this number from 2 to another, then you need to tweak Lemma 1 above. Lemma 1 was actually crafted specifically for the case $a = 2$.$\big)$ Having agreed all of that, the above sentence becomes

$$\text{“}\ \underline{\underline{\textit{When } 3 \textit{ and } s \textit{ are close, then } 2^3 \textit{ and } 2^s \textit{ are close.}}}\ \text{”}$$

The mathematically precise version of this sentence is Proposition 1 below $\big($the term ‘proposition’ refers to a statement whose significance within the paper is somewhere between ‘lemma’ and ‘theorem’$\big)$. I attach proof only for the completeness sake.

Proposition 1. No matter how large an integer $N\,\big(>0\big)$ you choose, you can find a rational number s (i) above 3 and (ii) below 3 each, such that the distance between 2^3 and 2^s is less than $\dfrac{1}{N}$.

Proof $\big($ends at the end of page 18 — follow it only if you have a strong stomach$\big)$**.**

(i) Set $s = 3 + \dfrac{1}{8N}$. Then

$$\begin{aligned} 2^s - 2^3 &= 2^3\left(2^{s-3} - 1\right) \\ &= 8\left(2^{\frac{1}{8N}} - 1\right) \\ &< 8\left(\frac{1}{8N} + 1 - 1\right) \qquad \big(\text{by Lemma 1}\big) \\ &= 8 \cdot \frac{1}{8N} = \frac{1}{N}. \end{aligned}$$

In short,

$$2^s - 2^3 < \frac{1}{N}.$$

(ii) Similarly, set $s = 3 - \dfrac{1}{8\,N}$. Then

$$
\begin{aligned}
2^3 - 2^s &= 2^3 \left(1 - \frac{1}{2^{3-s}} \right) \\
&= 8 \left(1 - \frac{1}{2^{\frac{1}{8N}}} \right) \\
&< 8 \left(1 - \frac{1}{\dfrac{1}{8\,N} + 1} \right) \qquad \Big(\text{by Lemma 1}\Big) \\
&= 8 \left(1 - \frac{8\,N}{1 + 8\,N} \right) \\
&= 8 \cdot \frac{1}{1 + 8\,N} \\
&< 8 \cdot \frac{1}{8\,N} = \frac{1}{N}.
\end{aligned}
$$

In short,

$$2^3 - 2^s < \frac{1}{N}.$$

Now our proof of Proposition 1 is complete. $\square$

- **Continuity of a^r as a function on r.**

We arrive at the final stage of today's lecture. We state the following theorem, which is the precise version of the statement " When r and s are close, then a^r and a^s are close."

Theorem 1. Let r be a positive rational number, and fixed. Let a be a positive real number $\big($not necessarily a rational number$\big)$, and fixed. Then the following conclusion holds:

No matter how large an integer $N\big(> 0\big)$ you choose, you can find a rational number s (i) above r and (ii) below r each, such that the distance between a^r and a^s is less than $\dfrac{1}{N}$.

You are going to see how this has a bearing on our business, the exponential functions, in the forthcoming lectures. — To be continued.