Math 105 TOPICS IN MATHEMATICS
REVIEW OF LECTURES – XVII

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§17. Binary system. Base ℓ system.

- Near the end of the last lecture, I threw

$$2^{\sqrt{2}}.$$

I asked: "What does this mean?" True, this is

$$2^{1.4142135623730950488016887242096980785696718753769...},$$

based on

$$\sqrt{2} \ = \ 1.4142135623730950488016887242096980785696718753769... \ .$$

But the problem is that this decimal expression for $\sqrt{2}$ continues endlessly. When we defined the rational number exponent, we substantially relied on the fact that a rational number can be written in the form

$$\frac{n}{k} \qquad \Big(n, \ k \ : \ \text{integers}\Big).$$

We have defined $\boxed{a^{\frac{n}{k}}}$ as the k-th root of the n-th power of a (or, the same to say, the n-th power of the k-th root of a). But we cannot apply that when the exponent is $\sqrt{2}$ because $\sqrt{2}$ is *irrational*, hence $\sqrt{2}$ cannot be written as $\dfrac{n}{k}$ with two integers n and k. We have already seen the proof of this fact, that $\sqrt{2}$ is irrational, in "Review of Lectures – XIII". Without such expression, we seemingly do not have a way to proceed. So sounds like we are doomed. Is there anthing we can do?

- **Idea.** Yes, try the following:

$$
\begin{aligned}
&2^{1.4} \quad , \\
&2^{1.41} \quad , \\
&2^{1.414} \quad , \\
&2^{1.4142} \quad , \\
&2^{1.41421} \quad , \\
&2^{1.414213} \quad , \\
&2^{1.4142135} \quad , \\
&2^{1.41421356} \quad , \\
&2^{1.414213562} \quad , \\
&2^{1.4142135623} \quad , \\
&2^{1.41421356237} \quad , \\
&2^{1.414213562373} \quad , \\
&2^{1.4142135623730} \quad , \\
&\quad\vdots
\end{aligned}
$$

The exponents are the truncations of the decimal expression of $\sqrt{2}$, so they are all rational numbers. So each of these lines makes perfect sense. Now, most importantly, from "Review of Lectures – XV" we have learned that, when the two exponents are close, 2-to-the-power of those exponents are also close. So the gap between two adjacent ones in the above list are getting smaller and smaller as you move down. So, we are naturally led to define $2^{\sqrt{2}}$ as follows:

Definition. For each positive integer n, let d_n be the truncation at the n-th place under the decimal point of the decimal expression of $\sqrt{2}$. Thus d_n is a rational number for each n. Now, we define $2^{\sqrt{2}}$ as the 'limit'

$$\boxed{2^{\sqrt{2}} \quad = \quad \lim_{n \to \infty} 2^{d_n}} \quad .$$

⋆ Now, we must address whether such a limit really exists (Question 1, page 16). Below is what the computer shows. You can see that the digits get stagnant:

$$
\begin{aligned}
2^{1.4} &= 2.63901582154578851874800394246\ldots\ ,\\
2^{1.41} &= 2.65737162819302316196830322510\ldots\ ,\\
2^{1.414} &= 2.66474965018404354228052965985\ldots\ ,\\
2^{1.4142} &= 2.66511908853235146915658056556\ldots\ ,\\
2^{1.41421} &= 2.66513756179419556891563407046\ldots\ ,\\
2^{1.414213} &= 2.66514310379771798925398518573\ldots\ ,\\
2^{1.4142135} &= 2.66514402746609214186201614377\ldots\ ,\\
2^{1.41421356} &= 2.66514413830631855217766568697\ldots\ ,\\
2^{1.414213562} &= 2.66514414200099284524439803529\ldots\ ,\\
2^{1.4142135623} &= 2.66514414255519398964617229213\ldots\ ,\\
2^{1.41421356237} &= 2.66514414268450759002316845653\ldots\ ,\\
2^{1.414213562373} &= 2.66514414269004960146803707575\ldots\ ,\\
2^{1.4142135623730} &= 2.66514414269004960146803707575\ldots\ ,\\
2^{1.41421356237309} &= 2.66514414269021586181138331238\ldots\ ,\\
2^{1.414213562373095} &= 2.66514414269022509849712477027\ldots\ ,\\
2^{1.4142135623730950} &= 2.66514414269022509849712477027\ldots\ ,\\
2^{1.41421356237309504} &= 2.66514414269022517239061070194\ldots\ ,\\
2^{1.414213562373095048} &= 2.66514414269022518716930788827\ldots\ ,\\
2^{1.4142135623730950488} &= 2.66514414269022518864717760690\ldots\ ,\\
2^{1.41421356237309504880} &= 2.66514414269022518864717760690\ldots\ ,\\
2^{1.414213562373095048801} &= 2.66514414269022518864902494405\ldots\ ,\\
2^{1.4142135623730950488016} &= 2.66514414269022518865013334634\ldots\ ,\\
2^{1.41421356237309504880168} &= 2.66514414269022518865028113331\ldots\ ,\\
2^{1.414213562373095048801688} &= 2.66514414269022518865029591201\ldots\ ,
\end{aligned}
$$

- **Binary system.**

But then you might suggest that the above is 'artificial'. If you are a computer geek, you know your computer only uses 0 and 1 to perform. What does that mean? Does that mean it does not recognize 2, 3, 4, *etc.*? Or does that mean that it only considers numbers that are made of 0 and 1, such as 11, 100, 110, *etc.*, but precludes any other number? No, not really. It crunches all numbers, of course. It is just that there is a way to express all numbers only using 0 and 1. What is that supposed to mean? To make sense of this, it is good to remind ourselves how we usually express numbers in decimals: We only need ten different arabic numerals

$$0, \ 1, \ 2, \ 3, \ 4, \ 5, \ 6, \ 7, \ 8 \ \text{and} \ 9$$

to describe numbers, where the idea is that what's next to 9 $\big($the 'successor' to 9$\big)$ is 10. And yes, literally, I just wrote 10 as 1 and 0 in juxtaposition, and 1 and 0 are already in the above list of the arabic numerals. Furthermore, what's next to 10 $\big($the 'successor' to 10$\big)$ is 11, where, once again, I just wrote it as 1 and 1 in juxtaposition, and so on, so we won't need any more numeral outside of the above list no matter how far you go. But strictly from a mathematical standpoint, there is no logical reason that the number of numerals has to be ten. Though this may sound odd, we can live with only two arabic numerals, namely, 0 and 1, with the proviso that what's next to 1 $\big($the 'successor' to 1$\big)$ is 10, then next to that $\big($the 'successor' to 10$\big)$ is 11, then next to that $\big($the 'successor' to 11$\big)$ is 100, and so forth. Watch how the numbers grow in that system:

$$\begin{aligned} 1,& \\ 10,& \\ 11,& \\ 100,& \\ 101,& \\ 110,& \\ 111,& \\ 1000,& \\ 1001,& \\ 1010,& \\ 1011,& \\ 1100,& \\ 1101,& \\ 1110,& \\ 1111,& \\ \vdots& \end{aligned}$$

So, what is written as 10 in this system actually means 2 (two), what is written as 11 in this system actually means 3 (three), what is written as 100 in this system actually means 4 (four), and so on. This way we can express numbers only using 1 and 0. This way of writing numbers is called

$$\text{“binary system” (“base 2 system”)}.$$

The usual way to write numbers is called

$$\text{“decimal system” (“base 10 system”)}.$$

Here is the comparison:

(in binary)		(in decimal)
1	=	1
10	=	2
11	=	3
100	=	4
101	=	5
110	=	6
111	=	7
1000	=	8
1001	=	9
1010	=	10
1011	=	11
1100	=	12
1101	=	13
1110	=	14
1111	=	15
10000	=	16
10001	=	17
10010	=	18
10011	=	19
10100	=	20
	⋮	

• Agree that 10000....0 written in the binary system (1 followed by 0s) is a 2-to-the power. Also agree that 11111....1 written in the binary system (staright 1s) is a Mersenne number (see "Review of Lectures – VIII"). For example:

Example 1. (in binary) (in decimal)

$$
\begin{aligned}
10 &= 2^1 \\
100 &= 2^2 \\
1000 &= 2^3 \\
10000 &= 2^4 \\
100000 &= 2^5 \\
1000000 &= 2^6 \\
10000000 &= 2^7 \\
100000000 &= 2^8 \\
1000000000 &= 2^9 \\
10000000000 &= 2^{10} \\
100000000000 &= 2^{11} \\
1000000000000 &= 2^{12} \\
&\vdots
\end{aligned}
$$

Example 2. (in binary) (in decimal)

$$
\begin{aligned}
1 &= 2^1 - 1 \\
11 &= 2^2 - 1 \\
111 &= 2^3 - 1 \\
1111 &= 2^4 - 1 \\
11111 &= 2^5 - 1 \\
111111 &= 2^6 - 1 \\
1111111 &= 2^7 - 1 \\
11111111 &= 2^8 - 1 \\
111111111 &= 2^9 - 1 \\
1111111111 &= 2^{10} - 1 \\
11111111111 &= 2^{11} - 1 \\
111111111111 &= 2^{12} - 1 \\
&\vdots
\end{aligned}
$$

Example 3. You remember that the largest known prime (as of December 2014) is a Mersenne prime, and it is

$$2^{57885161} - 1$$

(from "Review of Lectures – VIII"). The above is written in the usual decimals. But if you are to convert this into the binary system, it is

$$\underbrace{1\,1\,1\,1\,1\ \ldots\ 1}_{57885161}$$

(57885161 straight 1s).

Exercise 1. Convert each of the following expression of numbers in the binary system back into the usual decimal system.

(1) 10000. (2) 10101. (3) 11111. (4) 100100.

(5) 11111111.

[**Answers**]: (1) 16. (2) 21. (3) 31. (4) 36.

(5) 255.

Exercise 2. Convert each of the following expressions of numbers in the usual decimal system into the binary system.

(1) 17. (2) 19. (3) 25. (4) 27.

(5) 100.

[**Answers**]: (1) 10001. (2) 10011. (3) 11001. (4) 11011.

(5) 1100100.

- **Hexadecimal systyem.**

Now, in the same manner, we can consider

$$\text{“hexadecimal system”} \ \Big(\text{“base 16 system”}\Big).$$

Thus we supply sixteen arabic numerals. Since in our real world we only have ten of those, so we make up another six:

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ A,\ B,\ C,\ D,\ E \text{ and } F.$$

In this system, here is how the number grows (from left to right in the first line, then from left to right in the second line, and so on):

	1,	2,	3,	4,	5,	6,	7,	8,	9,	A,	B,	C,	D,	E,	F,
10,	11,	12,	13,	14,	15,	16,	17,	18,	19,	1A,	1B,	1C,	1D,	1E,	1F,
20,	21,	22,	23,	24,	25,	26,	27,	28,	29,	2A,	2B,	2C,	2D,	2E,	2F,
30,	31,	32,	33,	34,	35,	36,	37,	38,	39,	3A,	3B,	3C,	3D,	3E,	3F,
40,	41,	42,	43,	44,	45,	46,	47,	48,	49,	4A,	4B,	4C,	4D,	4E,	4F,
50,	51,	52,	53,	54,	55,	56,	57,	58,	59,	5A,	5B,	5C,	5D,	5E,	5F,
60,	61,	62,	63,	64,	65,	66,	67,	68,	69,	6A,	6B,	6C,	6D,	6E,	6F,
70,	71,	72,	73,	74,	75,	76,	77,	78,	79,	7A,	7B,	7C,	7D,	7E,	7F,
80,	81,	82,	83,	84,	85,	86,	87,	88,	89,	8A,	8B,	8C,	8D,	8E,	8F,
90,	91,	92,	93,	94,	95,	96,	97,	98,	99,	9A,	9B,	9C,	9D,	9E,	9F,
A0,	A1,	A2,	A3,	A4,	A5,	A6,	A7,	A8,	A9,	AA,	AB,	AC,	AD,	AE,	AF,
B0,	B1,	B2,	B3,	B4,	B5,	B6,	B7,	B8,	B9,	BA,	BB,	BC,	BD,	BE,	BF,
C0,	C1,	C2,	C3,	C4,	C5,	C6,	C7,	C8,	C9,	CA,	CB,	CC,	CD,	CE,	CF,
D0,	D1,	D2,	D3,	D4,	D5,	D6,	D7,	D8,	D9,	DA,	DB,	DC,	DD,	DE,	DF,
E0,	E1,	E2,	E3,	E4,	E5,	E6,	E7,	E8,	E9,	EA,	EB,	EC,	ED,	EE,	EF,
F0,	F1,	F2,	F3,	F4,	F5,	F6,	F7,	F8,	F9,	FA,	FB,	FC,	FD,	FE,	FF,
100,	101,	102,	103,	104,	105,	106,	107,	108,	109,	10A,	10B,	10C,	10D,	10E,	10F,
110,	$\cdots$														

Here is the comparison:

(in hexadecimal)		(in decimal)
1	=	1
2	=	2
3	=	3
4	=	4
5	=	5
6	=	6
7	=	7
8	=	8
9	=	9
A	=	10
B	=	11
C	=	12
D	=	13
E	=	14
F	=	15
1 0	=	16
1 1	=	17
1 2	=	18
1 3	=	19
1 4	=	20
1 5	=	21
1 6	=	22
1 7	=	23
1 8	=	24
1 9	=	25
$1A$	=	26
$1B$	=	27
$1C$	=	28
$1D$	=	29
$1E$	=	30
$1F$	=	31
2 0	=	32
	⋮	

• Agree that 10000....0 written in the hexiadecimal system (1 followed by 0s) is a 16-to-the power. Also agree that $FFFFF....F$ written in the hexadecimal system (staright Fs) is a 16-to-the-power minus 1. For example:

Example 3. (in hexadecimal) (in decimal)

$$
\begin{array}{rcl}
10 & = & 16^1 \\
100 & = & 16^2 \\
1000 & = & 16^3 \\
10000 & = & 16^4 \\
100000 & = & 16^5 \\
1000000 & = & 16^6 \\
10000000 & = & 16^7 \\
100000000 & = & 16^8 \\
1000000000 & = & 16^9 \\
10000000000 & = & 16^{10} \\
100000000000 & = & 16^{11} \\
1000000000000 & = & 16^{12} \\
 & \vdots &
\end{array}
$$

Example 4. (in hexadecimal) (in decimal)

$$
\begin{array}{rcl}
F & = & 16^1 - 1 \\
FF & = & 16^2 - 1 \\
FFF & = & 16^3 - 1 \\
FFFF & = & 16^4 - 1 \\
FFFFF & = & 16^5 - 1 \\
FFFFFF & = & 16^6 - 1 \\
FFFFFFF & = & 16^7 - 1 \\
FFFFFFFF & = & 16^8 - 1 \\
FFFFFFFFF & = & 16^9 - 1 \\
FFFFFFFFFF & = & 16^{10} - 1 \\
FFFFFFFFFFF & = & 16^{11} - 1 \\
FFFFFFFFFFFF & = & 16^{12} - 1 \\
 & \vdots &
\end{array}
$$

⋆ You remember that I briefly mentioned that a Babylonian clay tablet contained an approximation of $\sqrt{2}$ (“Review of Lectures – XIII”). That approximation actually is the first few digits of $\sqrt{2}$ expressed in base 60 system, or the <u>sexagesimal system</u> .

Exercise 3. Convert each of the following expression of numbers in the hexadecimal system back into the usual decimal system.

(1) F. (2) $3B$. (3) $A0$. (4) CC.

(5) 100. (6) $A0A$. (7) $2C00$. (8) 10001.

[**Answers**]: (1) 15. (2) 59. (3) 160. (4) 204.

(5) 256. (6) 2570. (7) 11264. (8) 65537.

Exercise 4. Convert each of the following expressions of numbers in the usual decimal system into the hexadecimal system.

(1) 13. (2) 17. (3) 28. (4) 46.

(5) 112. (6) 255. (7) 1000.

[**Answers**]: (1) D. (2) 11. (3) $1C$. (4) $2E$.

(5) 70. (6) FF. (7) $3E8$.

• The following is $\sqrt{2}$ expressed in the binary, the decimal, and the hexiadecimal, systems:

1.011010100000100111100110011001111111001110111100... (<u>in binary</u>).

1.4142135623730950488016887... (<u>in decimals</u>).

$1.6A09E667F3BC$... (<u>in hexadecimals</u>).

Earlier I said some of you might argue that what's on page 3 is 'artificial'. Indeed, it uses the truncations of the base 10 expression of $\sqrt{2}$. Actually, nothing stops us from considering the truncations of the base ℓ expression of $\sqrt{2}$ with all different ℓ. So, for example, we may use the base $\ell = 16$ to do the same thing as above. Namely, we use the sequence

$$
\begin{array}{l}
1.6 \\
1.6A \\
1.6A0 \\
1.6A09 \\
1.6A09E \\
1.6A09E6 \\
1.6A09E66 \\
1.6A09E667 \\
1.6A09E667F \\
1.6A09E667F3 \\
1.6A09E667F3B \\
1.6A09E667F3BC \\
\vdots
\end{array}
$$

For the sake of avoiding confusion, convert them back to the usual decimal system:

$$
\begin{array}{lcl}
\text{(in hexadecimal)} & & \text{(in decimal)} \\
1.6 & = & 1.375, \\
1.6A & = & 1.4140625, \\
1.6A0 & = & 1.4140625, \\
1.6A09 & = & 1.4141998291015625, \\
1.6A09E & = & 1.4142131805419921875, \\
1.6A09E6 & = & 1.41421353816986083984375, \\
1.6A09E66 & = & 1.414213560521602630615234375, \\
1.6A09E667 & = & 1.41421356215141713619232177734375, \\
1.6A09E667F & = & 1.414213562369695864617824554443359375, \\
1.6A09E667F3 & = & 1.4142135623724243487231433391571044921875, \\
1.6A09E667F3B & = & 1.41421356237304962633061222732067108154296875, \\
1.6A09E667F3BC & = & 1.4142135623730922588947578333318233489990234375, \\
 & \vdots &
\end{array}
$$

Next let's form 2-raised-to-the-power-of each of these. Let the computers handle it: (numbers on this page are all in the usual decimals):

$$2^{1.375} = 2.5936791093020193318675082355584... ,$$

$$2^{1.4140625} = 2.6648650941663228987032867589826... ,$$

$$2^{1.4140625} = 2.6648650941663228987032867589826... ,$$

$$2^{1.4141998291015625} = 2.6651187728283058488922292702281... ,$$

$$2^{1.4142131805419921875} = 2.6651434373195378433478057823147... ,$$

$$2^{1.4142135381698608398484375} = 2.6651440979786919006900352425025... ,$$

$$2^{1.41421356052160263061523437 5} = 2.6651441392698944669489994320222... ,$$

$$2^{1.4142135621514171361923217773437 5} = 2.6651441422807113457627407613288... ,$$

$$2^{1.414213562369695864617824554443359375} = 2.6651441426839457494335671629988... ,$$

$$2^{1.41421356237242434872314333915710449218 75} = 2.6651441426889861794798385660844... ,$$

$$2^{1.41421356237304962633061222732067108154296875} = 2.6651441426901412780321104385582... ,$$

$$2^{1.4142135623730922588947578333318233489990234375} = 2.6651441426902200347515835390044... ,$$

The previous page is crammed because I wrote $2^{1.414213\cdots}$ on the left-hand side in each line. But what we want to see is how the digits on the right-hand side get stagnant. So let me show you only the right-hand sides. Below I give you twenty four lines, namely, up to the 24-th place truncation under the hexadecimal point of $\sqrt{2}$ (the previous page only shows the first twelve of them). The digits below are re-expressed in the usual decimal system:

$$
\begin{aligned}
&2.5936791093020193318675082355 84\ldots \quad ,\\
&2.6648650941663228987032867589 82\ldots \quad ,\\
&2.6648650941663228987032867589 82\ldots \quad ,\\
&2.6651187728283058488922292702 28\ldots \quad ,\\
&2.6651434373195378433478057823 14\ldots \quad ,\\
&2.6651440979786919006900352425 02\ldots \quad ,\\
&2.6651441392698944669489994320 22\ldots \quad ,\\
&2.6651441422807113457627407613 28\ldots \quad ,\\
&2.6651441426839457494335671629 98\ldots \quad ,\\
&2.6651441426889861794798385660 84\ldots \quad ,\\
&2.6651441426901412780321104385 58\ldots \quad ,\\
&2.6651441426902200347515835390 04\ldots \quad ,\\
&2.6651441426902249570465506078 58\ldots \quad ,\\
&2.6651441426902251877791271892 12\ldots \quad ,\\
&2.6651441426902251877791271892 12\ldots \quad ,\\
&2.6651441426902251885802819690 08\ldots \quad ,\\
&2.6651441426902251886491312078 96\ldots \quad ,\\
&2.6651441426902251886499135856 12\ldots \quad ,\\
&2.6651441426902251886502803251 64\ldots \quad ,\\
&2.6651441426902251886502971340 62\ldots \quad ,\\
&2.6651441426902251886502972295 66\ldots \quad ,\\
&2.6651441426902251886502972474 74\ldots \quad ,\\
&2.6651441426902251886502972497 12\ldots \quad ,\\
&2.6651441426902251886502972498 52\ldots \quad ,\\
&\qquad\qquad\vdots
\end{aligned}
$$

• So, once again, the digits get stagnant. Based on this observation you may want to counteroffer me the following alternative definition of $2^{\sqrt{2}}$:

Alternative Definition $\big($use $\boldsymbol{\ell} = 16\big)$.

For each positive integer n, let h_n be the truncation at the n-th place under the hexadecimal point of the hexadecimal expression of $\sqrt{2}$, re-expressed in the decimal system. Thus

$$h_1 = 1.375 \qquad \Big(\text{originally } 1.6 \text{ in hexadecimals}\Big),$$

$$h_2 = 1.4140625 \qquad \Big(\text{originally } 1.6A \text{ in hexadecimals}\Big),$$

$$h_3 = 1.4140625 \qquad \Big(\text{originally } 1.6A0 \text{ in hexadecimals}\Big),$$

$$h_4 = 1.4141998291015625 \qquad \Big(\text{originally } 1.6A09 \text{ in hexadecimals}\Big),$$

$$h_5 = 1.4142131805419921875 \qquad \Big(\text{originally } 1.6A09E \text{ in hexadecimals}\Big),$$

$$h_6 = 1.414213538169860839843 75$$

$$\Big(\text{originally } 1.6A09E6 \text{ in hexadecimals}\Big),$$

$$\vdots$$

These are all rational numbers. What's in the previous page is nothing else but $\big($the usual decimal expressions of$\big)$

$$2^{h_1},\ 2^{h_2},\ 2^{h_3},\ 2^{h_4},\ 2^{h_5},\ 2^{h_6},\ 2^{h_7},\ 2^{h_8},\ 2^{h_9},\ 2^{h_{10}},\ 2^{h_{11}},\ 2^{h_{12}},$$

$$2^{h_{13}},\ 2^{h_{14}},\ 2^{h_{15}},\ 2^{h_{16}},\ 2^{h_{17}},\ 2^{h_{18}},\ 2^{h_{19}},\ 2^{h_{20}},\ 2^{h_{21}},\ 2^{h_{22}},\ 2^{h_{23}},\ 2^{h_{24}},$$

listed in this order. Now, we define $2^{\sqrt{2}}$ as the 'limit' of this last sequence:

$$\boxed{2^{\sqrt{2}} = \lim_{n \to \infty} 2^{h_n}} \ .$$

So $2^{\sqrt{2}}$ is the number carrying the stagnated digits in the previous page.

- **The sequence $c_n(\ell)$ — truncation of digits in ℓ-ary expression of $\sqrt{2}$.**

Now, similarly, nothing stops you from choosing any positive integer ℓ (with $\ell > 1$) you like, express $\sqrt{2}$ in the "ℓ-ary" (=base ℓ) system, truncate it at the n-th place under the "ℓ-ary point" (the equivalent counterpart of the *decimal* point), and convert it back to the usual, decimally expressed, number. Call it

$$\boxed{c_n(\ell).}$$

For example, for $\ell = 16$, $c_n(16)$ is just h_n above, whereas for $\ell = 10$, $c_n(10)$ is

$$\begin{aligned} c_1(10) &= 1.4 \\ c_2(10) &= 1.41 \\ c_3(10) &= 1.414 \\ c_4(10) &= 1.4142 \\ c_5(10) &= 1.41421 \\ c_6(10) &= 1.414213 \\ c_7(10) &= 1.4142135 \\ c_8(10) &= 1.41421356 \\ &\vdots \end{aligned}$$

But like I said, there is no logical reason we should stick with just $c_n(10)$ or $c_n(16)$. We should give $c_n(\ell)$ for other integers ℓ with $\ell > 1$ an equal "playing time". Then define $2^{\sqrt{2}}$ as the limit of the sequence

$$\lim_{n \longrightarrow \infty} 2^{c_n(\ell)}.$$

Now, the real questions we need to address here are as follows:

Question 1.

Do those limits for various different ℓ (including $\ell = 10$ and $\ell = 16$) exist?

Question 2.

Do those limits for various different ℓ coincide, if they indeed exist?

Our computer experimentation as shown above suggests the answer is 'yes, those limits exist, and they do coincide'. Indeed, from page 3 $\big($the decimal case, or, the $\ell = 10$ case$\big)$ and from page 14 $\big($the hexadecimal case, or, the $\ell = 16$ case$\big)$, the stagnated portion of the digits are

$$2.6651441426902251886502\ , \qquad \text{and}$$
$$2.665144142690225188650297249\ ,$$

respectively. These match. But that's too weak an evidence. Indeed, this is just for two different ℓs: $\ell = 10, 16$. What we have seen is the first 22, and 27, digits under the decimal point, respectively, seemingly get stagnated as n gets past 24. Those stagnated digits, at least for the first 22 digits of them, coincide, though. For this, you can certainly do more using computer, going for more digits within $\ell = 10$ and $\ell = 16$ and also test with various different ℓ. But the bottom line you cannot cover all the infinitely many cases of ℓ and infinitely many digits with the computer, even if the accuracy of the computer can be trusted. Is there a theoretical way to prove that the limit in the above definition does not depend on the choice of ℓ?

The answers to Question 1 and Question 2 are:

Answer to Question 1. 'yes, the limit

$$\lim_{n \to \infty} 2^{c_n(\ell)}$$

exists for every ℓ.'

Answer to Question 2. 'yes, those limits

$$\lim_{n \to \infty} 2^{c_n(\ell)}$$

for different ℓs indeed coincide.'

But how do we know these answers? I will give you options.

Option A. Accept these answers, and move on.

Option B (recommended for math-bound students).

Read carefully the rest of this set of lecture notes, where you will find justifications for the above answers. Use my Office Hours for any questions you have.

• **Attention (!)** The following material is very abstract. This is about 'continuity'. A couple of lectures ago, I alluded that sooner or later the problem of continuity will haunt you, as in you cannot avoid it whichever directions you steer in the realm of math, it will follow you around. Anything beyond this paragraph (within this set of notes) are some 'extra curricular activity', so are not going to be on the exam, except we will freely use the notion of 'limits' and 'convergence' for the rest of the semester. Technically, though, these are an absolutely critical piece of information the whole rest of the materials logically depend on. So the inclusion is totally warranted. The goal is to clarify that the consistency of our definition cannot be seen without resorting to the Axiom of Continuity. What I was dwelling on back in "Review of Lectures – XV", which some of you might have thoughtlessly rebuffed as 'tangents', suddenly comes to life. Once again, the next material is optional. Please don't get too much stressed out trying to understand every inch of it, but do so only as long as your interest lasts. But if you are a math-major(-bound), then I encourage you to try to take time and thoroughly digest it.

• **Cauchy sequence.**

Let's consider an infinite sequence of (real) numbers which is monotonically increasing :

$$c_1,\ c_2,\ c_3,\ c_4,\ c_5,\ c_6,\ c_7,\ c_8,\ c_9,\ \cdots\ .$$

So each of $c_1,\ c_2,\ \cdots$ is a real number, and they satisfy

$$c_1 \le c_2 \le c_3 \le c_4 \le c_5 \le c_6 \le c_7 \le c_8 \le c_9 \le \cdots\ .$$

Here ' $\le$ ' means either ' $<$ ' or ' $=$ ' . We often use a short form to denote the sequence as follows:

$$\boxed{\left\{ c_n \right\}_n}$$

Here, n is called <u>the index</u>. An example of such $\left\{ c_n \right\}_n$ is $\left\{ c_n(10) \right\}_n$ defined in page 16:

$$
\begin{aligned}
c_1(10) &= 1.4 \\
c_2(10) &= 1.41 \\
c_3(10) &= 1.414 \\
c_4(10) &= 1.4142 \\
c_5(10) &= 1.41421 \\
c_6(10) &= 1.414213 \\
c_7(10) &= 1.4142135 \\
c_8(10) &= 1.41421356 \\
c_9(10) &= 1.414213562 \\
c_9(10) &= 1.4142135623 \\
&\vdots
\end{aligned}
$$

But that is just one example. In what follows we consider just any sequence $\left\{ c_n \right\}_n$ satisfying the monotonically increasing condition as above.

Below we are going to introduce the term

"Cauchy sequence."

Definition. A monotonically increasing sequence $\left\{ c_n \right\}_n$ is said to be a <u>Cauchy sequence</u>, if the following condition is satisfied:

Let N be an arbitrary positive integer. Then there exists an integer $q(N)$ such that, provided the index n is greater than $q(N)$, one has

$$c_n - c_{q(N)} < \frac{1}{N}.$$

A couple of lemmas:

Lemma 1. Let ℓ be an arbitrary positive integer, $\ell > 1$. Then $\left\{ c_n(\ell) \right\}_n$ defined in page 16 is a Cauchy sequence.

Proof. Let N be a positive integer. Let $q(N)$ be the smallest positive integer such that $\ell^{q(N)} > N$. As long as the index n exceeds $q(N)$ the quantity $c_n - c_{q(N)}$ is written in base ℓ as

$$0\,.\,0\,0\,0\,0\,...\,0\,0\,0 * * *$$

where the first $q(N)$ straight digits under the "ℓ-ary" point are 0, and $*$ are some digits. (For example, $c_8(10) - c_5(10) = 1.41421356 - 1.41421 = 0.00000356.$) This is less than

$$0\,.\,0\,0\,0\,0\,...\,0\,0\,1,$$

where there are $q(N) - 1$ straight 0s under the "ℓ-ary" point, and that is followed by 1. (This is just like $0.00000356 < 0.00001.$) This latter quantity is formally written as $\frac{1}{\ell^{q(N)}}$. So, in short,

$$c_n - c_{q(N)} < \frac{1}{\ell^{q(N)}} < \frac{1}{N}. \quad \square$$

Lemma 2. Let $\left\{ c_n \right\}_n$ be a monotonically increasing Cauchy sequence. Then there exists an integer C which none of c_n can exceed.

Proof. In the definition of Cauchy sequences, choose N to be 1. So $N = 1$. Then, there exists an index $q(1)$ such that, as long as the index n exceeds $q(1)$ we have $c_n - c_{q(1)} < 1$. This is the same as

$$c_n < c_{q(1)} + 1.$$

Now the smallest integer C that exceeds $c_{q(1)} + 1$ fulfills the requirement, namely, for any index n, (regardless of whether n exceeds $q(1)$ or not) c_n cannot exceed C. Indeed, by assumption, $\left\{ c_n \right\}_n$ is monotonically increasing, so none of c_1, c_2, c_3, $\cdots$, and $c_{q(1)-1}$ exceeds $c_{q(1)}$, and hence none of them exceeds C. $\square$

- **An upper-bound of a sequence.**

For a sequence $\left\{ c_n \right\}_n$, and for a (real) number C, if none of c_n can exceed C then such C is called an upper bound of the sequence $\left\{ c_n \right\}_n$. Using this terminology, we may paraphrase Lemma 2 as follows:

Lemma 2 paraphrased.

A monotonically increasing Cauchy sequence $\left\{ c_n \right\}_n$ has an upper-bound C.

Now, below is the key theorem.

Theorem 1 (continuity of 2^x).

Suppose $\left\{ c_n \right\}_n$ is an arbitrary monotonically increasing Cauchy sequence of rational numbers. Then $\left\{ 2^{c_n} \right\}_n$ is a monotonically increasing Cauchy sequence (of real numbers). In particular, let $\left\{ c_n(\ell) \right\}_n$ be as defined in page 16. Then $\left\{ 2^{c_n(\ell)} \right\}_n$ is a monotonically increasing Cauchy sequence (of real numbers).

Proof. By Lemma 2, $\left\{ c_n \right\}_n$ has an upper-bound C. We may assume that C is a positive integer. By assumption, for an arbitrary positive integer N, there exists an integer $q(N)$ such that, provided the index n exceeds $q(N)$, one has

$$c_n - c_{q(N)} < \frac{1}{N}.$$

Now, substitute N with $2^C \cdot N$. Then

$$(*) \qquad c_n - c_{q(2^C \cdot N)} < \frac{1}{2^C \cdot N}.$$

This $(*)$ is true provided n exceeds $q(2^C \cdot N)$.

Accordingly:

$$2^{c_n} - 2^{c_{q(2^C \cdot N)}} = 2^{c_{q(2^C \cdot N)}} \left(2^{c_n - c_{q(2^C \cdot N)}} - 1 \right)$$

$$< 2^{c_{q(2^C \cdot N)}} \left(2^{\frac{1}{2^C N}} - 1 \right)$$

$$\Big(\text{by } (*) \text{ and the fact that } r < s \text{ implies } 2^r < 2^s\Big)$$

$$< 2^{c_{q(2^C \cdot N)}} \left(\frac{1}{2^C \cdot N} + 1 - 1 \right)$$

$$\Big(\text{by "Review of Lectures – XV", Lemma 1, page 14}\Big)$$

$$= 2^{c_{q(2^C \cdot N)}} \cdot \frac{1}{2^C \cdot N}$$

$$= \frac{2^{c_{q(2^C N)} - C}}{N}$$

$$< \frac{1}{N}$$

$$\Big(\text{by the fact that } C \text{ is an upper-bound of } \big\{ c_n \big\}_n \Big).$$

Now the proof of Theorem 1 is complete. □

- **The precise definition of a limit.**

So far we took the stance we are already familiar with the concept of 'limits'. (In other words, we 'pretended' we knew about 'limits'.) We have actually never given the mathematically precise definition of limits. So I hereby give you one. Since we only need it for a monotonically increasing sequence, so I will formulate it only for a monotonically increasing sequence. (The same remark applies to our definition of Cauchy sequence above.) Along the way the following technical terms are also introduced:

"convergence" (n), "convergent" (adj), "converge" (v),

and their antonyms

"divergence" (n), "divergent" (adj), "diverge" (v).

Definition. Let $\{c_n\}_n$ be a monotonically increasing sequence. We do not assume that it is a Cauchy sequence. Let α be a (real) number. We say $\{c_n\}_n$ <u>is convergent to</u> α (or, it <u>converges to</u> α) if

(i) $c_n \leq \alpha$ for any member c_n of the sequence, and moreover

(ii) if for an arbitrary positive integer N, there exists an integer $q(N)$ such that, provided the index n exceeds $q(N)$, one has

$$\alpha - c_n < \frac{1}{N}.$$

When $\{c_n\}_n$ is convergent to a real number α, we write

$$\lim_{n \to \infty} c_n = \alpha.$$

We say α is <u>the limit</u> of the sequence $\{c_n\}_n$ (as n goes to ∞). If there does not exist such real number α, then we say $\{c_n\}_n$ is <u>divergent</u> (or say it <u>diverges</u>).

Example. $\left\{ c_n(\ell) \right\}_n$ as defined in page 16 clearly converges to $\sqrt{2}$:

$$\lim_{n \to \infty} c_n(\ell) = \sqrt{2}.$$

Remark. By definition, the limit of any monotonically increasing sequence $\left\{ c_n \right\}_n$, provided it exists, is the smallest upper-bound of that sequence.

Next, let me state the axiom of continuity, which I touched in "Review of Lectures – XV" though I did not state it at that time:

Axiom of Continuity (Version 1).

For any Cauchy sequence $\left\{ c_n \right\}_n$ the limit

$$\lim_{n \to \infty} c_n$$

exists as a real number. (Version 2 is in page 26.)

So, in particular, by virtue of Lemma 1 in page 20 and Theorem 1 in page 21, the answer to Question 1 in page 16 is affirmative, namely, the limit

$$\lim_{n \to \infty} 2^{c_n(\ell)}$$

exists, for each integer ℓ with $\ell > 1$.

- **Can we 'prove' the Axiom of Continuity?**

Now, you might ask if we can prove the axiom of continuity. An excellent question. The truth is, no, not really. So, I said in math everything has to be proved. So, what's worng? The answer is, as much as it is mathematicians' second-nature to build everything from the scratch, there are certain things that needs to be assumed, as in no one can create something out of nothing. For instance, the existence of 0 and 1. We need to assume the existence of 0 and 1 or we cannot do anything. As for this, you may actually take the stance

"*Hey, I totally reject the existence of* 0 *and* 1."

What that entails is you would reject other numbers, because, as we have seen earlier, all other numbers are indeed made out of 0 and 1. Then your position is to reject all mathematics. So if, hypothetically, I took that position, then I would just quit lecturing, take everybody to the lake and play frysbee. Now, I am not bringing this up because I want to preach you some morality what's right and what's wrong. Not that. In fact, that is one acceptable stance, and probably that person would choose to live in the world of null, pay no heed to currency, clock, and all kinds of measurement, no sports because the winners are decided by scores, *etc.* Much less internet. A self-sufficient life style. I am not worrying about its morality. Rather, independently of that 'lifestyle' context, it makes sense to ask mathematically how much one can logically build up out of just 0 and 1, strictly within math? It has long been thought that everything within math can indeed be built out of just 0 and 1.

Some important discoveries were made in the 19-th and the 20-th centuries, which had overturned such belief. Namely, there turned out to be some postulations which mathematicians used to take for granted and which actually do not quite follow from whatever one can build out of 0 and 1. Those postulations are something which it is logically impossible to prove out of just 0 and 1. I can give you some analogy in geometry. The following is called 'parallel line axiom', or otherwise called 'Euclid's fifth postulate' (as it was so listed in Euclid's treatise '*Elements*'):

> "*Suppose a line, and a point off that line, are given. Then there is one and only one line that passes through the given point and is parallel to the given line.*"

As it turns out, no matter how hard you try, you cannot prove this parallel line axiom out of other geometric postulates which Euclid listed as undebatable and plain as the existence of 0 and 1. So in any case, you need to proclaim that you add this to your axiom list, and your math is going to be based on it. The Axiom of Continuity is another one of those postulations, you cannot prove it out of just 0 and 1 and whatever can be proved out of 0 and 1. Mathematicians describe this fact as the logical independence of the Axiom of Continuity from the arithmetic of numbers. So, in short, the Axiom of Continuity cannot be proven. That's why it is called an 'axiom'. In math, 'axioms' are something which you impose, and something you never prove. And of course, there are other axioms and sometimes one axiom can be used to prove another axiom. Then mathematicians say there is a logical dependence between those axioms. If Axiom B is proved from Axiom A and Axiom A is proved from Axiom B, then mathematicians say Axiom A and Axiom B are equivalent. You remember that I alluded that there are several equivalent versions of the Axiom of Continuity. Below let me highlight another version of the Axiom of Continuity, which is equivalent to the one already highlighted in the above:

Axiom of Continuity (Version 2).

Let $\left\{ c_n \right\}_n$ be a monotonically increasding sequence $\big($of real numbers$\big)$. Suppose $\left\{ c_n \right\}_n$ has an upper-bound C. Then $\left\{ c_n \right\}_n$ is convergent to some real number α. In other words, the limit

$$\lim_{n \to \infty} c_n = \alpha$$

exists as a real number.

Now, Lemma 2 $\big($in page 20, and its paraphrase in page 21$\big)$ says that any monotonically increasing Cauchy sequence has an upper-bound. Thus, Version 1 of Axiom of Continuity in page 24 immediately follows from the above Version 2. In other words, Version 2 is superficially stronger than Version 1. But in reality, those two versions of Axiom of Continuity are equivalent. In order to establish the equivalence of the two, you still need to prove the converse, namely, Version 2 follows from Version 1. For this, it suffices to prove that a monotonically increasing sequence that has an upper bound is a Cauchy sequence, without relying on either version of Axiom of Continuity. I will skip this. Also, I could have used Version 2 of Axiom of Continuity to answer Question 1, which would have been easier because the sequence $\left\{ 2^{c_n(\ell)} \right\}_n$ clearly has an upper-bound $2^2 = 4$. But instead I chose the path that I chose, namely, I chose to prove Theorem 1, and then applied Version 1 of Axiom of Continuity.

- It still remains to answer Question 2. For this we need the following lemma.

Lemma 3. Let $\left\{ c_n \right\}_n$ and $\left\{ d_n \right\}_n$ be two sequences, both monotonically increasing. Suppose both sequences are convergent:

$$\lim_{n \to \infty} c_n = \alpha, \qquad \lim_{n \to \infty} d_n = \beta.$$

Suppose for an arbitrary positive integer L there are infinitely many indices n satisfying

$$c_n - d_n < \frac{1}{L} \qquad \text{and} \qquad d_n - c_n < \frac{1}{L}.$$

Then $\alpha = \beta$.

Proof. The style (strategy) of the present proof is "proof by contradiction".

Without loss of generality, we may assume $\alpha \geq \beta$. So let us assume $\alpha > \beta$, to derive a contradiction. Choose a positive integer N such that

$$\frac{1}{N} < \frac{\alpha - \beta}{2}.$$

By the definition of limits, there exists a $q(N)$ such that, provided the index n exceeds $q(N)$ one has

$$\alpha - c_n < \frac{1}{N}.$$

Then

$$\begin{aligned} c_n &> \alpha - \frac{1}{N} \\ &> \alpha - \frac{\alpha - \beta}{2} \\ &= \frac{\alpha + \beta}{2}. \end{aligned}$$

In short,

$$c_n > \frac{\alpha + \beta}{2},$$

provided the index n exceeds $q(N)$. Meanwhile, β is an upper-bound of $\left\{ d_n \right\}_n$. Hence for n in the same range (namely, for n that exceeds $q(N)$) we have

$$\begin{aligned} c_n - d_n &\geq c_n - \beta \\ &> \frac{\alpha + \beta}{2} - \beta \\ &= \frac{\alpha - \beta}{2}. \end{aligned}$$

In short,

$$c_n - d_n > \frac{\alpha - \beta}{2},$$

for all n that exceeds $q\left(N\right)$. Note that $\dfrac{\alpha - \beta}{2}$ is a positive real number which does not depend on n. Hence this contradicts the assumption. □

Theorem 2. Let ℓ be an integer, with $\ell > 1$. Let $\left\{ c_n\left(\ell\right) \right\}_n$ be the sequence defined in page 16. Then

$$\lim_{n \to \infty} 2^{c_n\left(\ell\right)}$$

does not depend on ℓ.

Proof. Let ℓ and ℓ' be two integers, both greater than 1. Let

$$\lim_{n \to \infty} 2^{c_n\left(\ell\right)} = \alpha, \qquad \text{and} \qquad \lim_{n \to \infty} 2^{c_n\left(\ell'\right)} = \beta.$$

We will prove $\alpha = \beta$. As a starter, let's agree that either one of the following holds:

(i) $c_n\left(\ell\right) \leq c_n\left(\ell'\right)$ holds for an infinitely many n.

(ii) $c_n\left(\ell\right) \geq c_n\left(\ell'\right)$ holds for an infinitely many n.

((i) happens if $\ell \leq \ell'$, and (ii) happens if $\ell \geq \ell'$, but we don't need that information.) Let's assume (i) without loss of generality. Now, for n such that $c_n\left(\ell\right) \leq c_n\left(\ell'\right)$,

$$\left(0 \leq\right) c_n\left(\ell'\right) - c_n\left(\ell\right) \leq \sqrt{2} - c_n\left(\ell\right),$$

because $\sqrt{2}$ is clearly an upper-bound of $\left\{ c_n\left(\ell'\right) \right\}_n$. Now, we claim

$$\sqrt{2} - c_n\left(\ell\right) \leq \frac{1}{\ell^n}.$$

Indeed, $\sqrt{2} - c_n(\ell)$ is written in the "ℓ-ary" system as

$$0 \,.\, 0\,0\,0\,0 \ldots 0\,0\,0 * * *$$

where the first n straight digits under the "ℓ-ary" point are 0, and $*$ are some digits. This is less than

$$0 \,.\, 0\,0\,0\,0 \ldots 0\,0\,1,$$

where there are $n-1$ straight 0s under the "ℓ-ary" point, and that is followed by 1. The latter is formally written as $\frac{1}{\ell^n}$. Hence we conclude

$$(\#) \qquad \Big(0 \leq\Big)\ c_n(\ell') - c_n(\ell) \leq \frac{1}{\ell^n}.$$

Accordingly:

$$2^{c_n(\ell')} - 2^{c_n(\ell)} = 2^{c_n(\ell)} \left(2^{c_n(\ell') - c_n(\ell)} - 1\right)$$

$$\leq 2^{c_n(\ell)} \left(2^{\frac{1}{\ell^n}} - 1\right)$$

$$\Big(\text{by } (\#) \text{ and the fact that } r \leq s \text{ implies } 2^r \leq 2^s\Big)$$

$$< 2^{c_n(\ell)} \left(\frac{1}{\ell^n} + 1 - 1\right)$$

$$\Big(\text{by "Review of Lectures – XV", Lemma 1, page 14}\Big)$$

$$= 2^{c_n(\ell)} \cdot \frac{1}{\ell^n}$$

$$< 4 \cdot \frac{1}{\ell^n}$$

$$\Big(\text{by the fact that } 4 \text{ is one upper-bound of } \left\{2^{c_n(\ell)}\right\}_n\Big).$$

In short,

$$(\#\#) \qquad \Big(0 \;\leq\Big)\; 2^{\,c_n(\ell')} - 2^{\,c_n(\ell)} < \frac{4}{\ell^n}.$$

This holds for infinitely many n. Now, if another index m exceeds n then

$$\frac{4}{\ell^m} < \frac{4}{\ell^n}.$$

So (##) clearly implies that, for each n such that (##) holds, there are infinitely many m each of which exceeds n and moreover satisfies

$$(\#\#\#) \qquad \Big(0 \;\leq\Big)\; 2^{c_m(\ell')} - 2^{\,c_m(\ell)} < \frac{4}{\ell^n}.$$

In particular, the following holds: Let L be an arbitrary positive integer. Choose a large enough n such that both (##) and

$$\frac{4}{\ell^n} < \frac{1}{L}$$

hold. Then for such n there are infinitely many indices m such that

$$(\#\#\#) \qquad \Big(0 \;\leq\Big)\; 2^{c_m(\ell')} - 2^{\,c_m(\ell)} < \frac{1}{L}.$$

Hence Lemma 3 is applicable for $\Big\{ 2^{c_n(\ell)} \Big\}_n$ and $\Big\{ 2^{c_n(\ell')} \Big\}_n$. Namely, by Lemma 3, their limits are equal. Now the proof of Theorem 2 is complete. $\square$

Hence we have affirmatively answered Question 2 in page 16.

Definition $\left(\mathbf{2}^{\sqrt{\mathbf{2}}}\right)$**.**

Let ℓ be an arbitrary integer, with $\ell > 1$. Let $\left\{ c_n(\ell) \right\}_n$ be the sequence defined in page 16. We define $2^{\sqrt{2}}$ as

$$\boxed{2^{\sqrt{2}} \quad = \quad \lim_{n \to \infty} \; 2^{c_n(\ell)}.}$$

By Theorem 1, and Axiom of Continuity $($Version 1$)$, the limit indeed exists, and by Theorem 2, the limit does not depend on the choice of ℓ.

- Though we will not prove it, we have a stronger result:

Theorem 4. Let $\left\{ c_n \right\}_n$ be any monotonically increasing sequence of rational numbers such that

$$\lim_{n \to \infty} c_n \quad = \quad \sqrt{2}.$$

Then the limit

$$\lim_{n \to \infty} 2^{c_n}$$

exists, and it does not depend on the choice of $\left\{ c_n \right\}_n$. Hence, conforming to the above definition, the limit equals $2^{\sqrt{2}}$.

⋆ Phew. So, this is how we officially define the number $2^{\sqrt{2}}$.

- **Is $2^{\sqrt{2}}$ rational or irrational?**

Once there was a vexing problem: Decide whether $2^{\sqrt{2}}$ is rational or irrational. Unlike the fact that $\sqrt{2}$ is irrational, whose proof was elementary, this problem turns out to be super-duper difficult. The answer is actually known: $2^{\sqrt{2}}$ is irrational. It is a special case of a theorem proved by a mathematician named Kuzmin in 1930, and also of another theorem proved independently by two mathematicians named Gelfond, and Schneider, in 1934.* So let me highlight:

Theorem. $2^{\sqrt{2}}$ is irrational.

I am simply citing this theorem, but I am not going to explain how Gelfond and Schneider proved it. Actually, this belongs to a rather specialized area of math called 'transendental number theory'. Their proof is not something every working mathematician knows. The theorem is famous enough, so mathematicians working in other areas at least know the statement of the theorem.

- **Tetration.**

Finally, a food for thought. Consider a sequence $\left\{ c_n \right\}_n$ defined by

$$c_1 = \sqrt{2},$$

$$c_2 = \sqrt{2}^{\sqrt{2}},$$

$$c_3 = \sqrt{2}^{\left(\sqrt{2}^{\sqrt{2}}\right)},$$

$$c_4 = \sqrt{2}^{\left(\sqrt{2}^{\left(\sqrt{2}^{\sqrt{2}}\right)}\right)},$$

$$c_5 = \sqrt{2}^{\left(\sqrt{2}^{\left(\sqrt{2}^{\left(\sqrt{2}^{\sqrt{2}}\right)}\right)}\right)},$$

$$\vdots$$

*Rodion Kuzmin (1891–1949), Alexander Gelfond (1906–1968), Theodor Schneider (1911–1988).

Thus

$$c_n = \sqrt{2}^{\,c_{n-1}} .$$

First, c_1 is just $\sqrt{2}$. c_2 is the square-root of $2^{\sqrt{2}}$. As complicated as each individual c_n may be, surprisingly enough, this sequence converges to 2.

Extra credit homework. Prove

$$\lim_{n \to \infty} c_n = 2.$$

(This assignment might require you to know what I am yet to cover. So, no due date set yet.)

Now, nothing stops you from picking a different number (a real number greater than 1), your favorite number, call it a, and do the same:

$$c_1 = a,$$

$$c_2 = a^a,$$

$$c_3 = a^{(a^a)},$$

$$c_4 = a^{\left(a^{(a^a)}\right)},$$

$$c_5 = a^{\left(a^{\left(a^{(a^a)}\right)}\right)},$$

and so on. More generally,

$$c_n = a^{c_{n-1}}.$$

Such a sequence (or the procedure to define such a sequence) is called '<u>tetration</u>'. Now, clearly for $a = 2$ the sequence is divergent (explain why). So, there is a threshold number a between $\sqrt{2}$ and 2 that separates the convergence and divergence. The existence of such threshold number is actually known. It is $e^{\frac{1}{e}}$. Moreover, for $a = e^{\frac{1}{e}}$ the sequence is known to actually converge.