Math 290 ELEMENTARY LINEAR ALGEBRA

SOLUTION FOR EXTRA CREDIT HOMEWORK – I (09/13)

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[I] (20pts) (1) Consider the system

$$
(**)\qquad
\begin{array}{lll}
-x_1 - x_6 = 0, & x_4 - x_{10} = 0, & -x_3 - x_{14} = 0, \\
x_8 + x_9 = 0, & -x_7 + x_{13} = 0, & -x_{11} - x_{16} = 0, \\
x_3 - x_8 = 0, & -x_2 - x_{12} = 0, & -x_1 - x_{16} = 0, \\
-x_6 - x_{11} = 0, & -x_5 - x_{15} = 0, & -x_9 + x_{14} = 0, \\
-x_4 - x_7 = 0, & -x_1 - x_{11} = 0, & x_2 - x_{15} = 0, \\
-x_5 + x_{12} = 0, & x_6 + x_{16} = 0, & x_{10} + x_{13} = 0.
\end{array}
$$

Clearly, (∗∗) is a system of of linear equations. Clearly, (∗∗) is a <u>homogeneous</u> system.

(2) We are to solve the above system (∗∗). First, look at the six equations highlighted below:

$$
(**)\qquad
\begin{array}{lll}
\boxed{-x_1 - x_6 = 0}, & x_4 - x_{10} = 0, & -x_3 - x_{14} = 0, \\
x_8 + x_9 = 0, & -x_7 + x_{13} = 0, & \boxed{-x_{11} - x_{16} = 0}, \\
x_3 - x_8 = 0, & -x_2 - x_{12} = 0, & \boxed{-x_1 - x_{16} = 0}, \\
\boxed{-x_6 - x_{11} = 0}, & -x_5 - x_{15} = 0, & -x_9 + x_{14} = 0, \\
-x_4 - x_7 = 0, & \boxed{-x_1 - x_{11} = 0}, & x_2 - x_{15} = 0, \\
-x_5 + x_{12} = 0, & \boxed{x_6 + x_{16} = 0}, & x_{10} + x_{13} = 0.
\end{array}
$$

Those highlighted equations read

$$
\begin{array}{lll}
x_1 = -x_6, & x_{11} = -x_1, & x_{11} = -x_{16}, \\
-x_6 = x_{11}, & -x_{16} = x_6, & x_1 = -x_{16}.
\end{array}
$$

From $x_1 = -x_6$ and $-x_6 = x_{11}$, we have

$$x_1 = x_{11}.$$

This and $x_{11} = -x_1$ yield

$$x_1 = x_{11} = 0.$$

This, $x_1 = -x_6$, and $-x_{16} = x_6$ yield

$$x_6 = x_{16} = 0.$$

In short, we obtain

$$\Big(x_1, \quad x_6, \quad x_{11}, \quad x_{16} \Big) \quad = \quad \Big(0, \quad 0, \quad 0, \quad 0 \Big).$$

Second, look at the four equations highlighted below:

$$(**) \qquad \begin{array}{lll} -x_1 - x_6 = 0, & x_4 - x_{10} = 0, & -x_3 - x_{14} = 0, \\ x_8 + x_9 = 0, & -x_7 + x_{13} = 0, & -x_{11} - x_{16} = 0, \\ x_3 - x_8 = 0, & \boxed{-x_2 - x_{12} = 0}\,, & -x_1 - x_{16} = 0, \\ -x_6 - x_{11} = 0, & \boxed{-x_5 - x_{15} = 0}\,, & -x_9 + x_{14} = 0, \\ -x_4 - x_7 = 0, & -x_1 - x_{11} = 0, & \boxed{x_2 - x_{15} = 0}\,, \\ \boxed{-x_5 + x_{12} = 0}\,, & x_6 + x_{16} = 0, & x_{10} + x_{13} = 0. \end{array}$$

Those highlighted equations read

$$-x_5 = -x_{12}, \quad x_2 = -x_{12}, \quad -x_5 = x_{15}, \quad x_2 = x_{15}.$$

These yield

$$x_2 = -x_5 = -x_{12} = x_{15}.$$

In other words, using a parameter b,

$$\Big(x_2, \quad x_5, \quad x_{12}, \quad x_{15} \Big) \quad = \quad \Big(b, \quad -b, \quad -b, \quad b \Big).$$

Third, look at the four equations highlighted below:

$$(**)\qquad \begin{array}{lll} -x_1 - x_6 = 0, & x_4 - x_{10} = 0, & \boxed{-x_3 - x_{14} = 0}\,, \\ \boxed{x_8 + x_9 = 0}\,, & -x_7 + x_{13} = 0, & -x_{11} - x_{16} = 0, \\ \boxed{x_3 - x_8 = 0}\,, & -x_2 - x_{12} = 0, & -x_1 - x_{16} = 0, \\ -x_6 - x_{11} = 0, & -x_5 - x_{15} = 0, & \boxed{-x_9 + x_{14} = 0}\,, \\ -x_4 - x_7 = 0, & -x_1 - x_{11} = 0, & x_2 - x_{15} = 0, \\ -x_5 + x_{12} = 0, & x_6 + x_{16} = 0, & x_{10} + x_{13} = 0. \end{array}$$

Those highlighted equations read

$$x_3 = -x_{14}, \quad x_8 = -x_9, \quad x_3 = x_8, \quad -x_9 = -x_{14}.$$

These yield

$$x_3 = x_8 = -x_9 = -x_{14}.$$

In other words, using a parameter c,

$$\Big(x_3, \;\; x_8, \;\; x_9, \;\; x_{14}\Big) \;=\; \Big(c, \;\; c, \;\; -c, \;\; -c\Big).$$

Last, look at the four equations highlighted below:

$$(**)\qquad \begin{array}{lll} -x_1 - x_6 = 0, & \boxed{x_4 - x_{10} = 0}\,, & -x_3 - x_{14} = 0, \\ x_8 + x_9 = 0, & \boxed{-x_7 + x_{13} = 0}\,, & -x_{11} - x_{16} = 0, \\ x_3 - x_8 = 0, & -x_2 - x_{12} = 0, & -x_1 - x_{16} = 0, \\ -x_6 - x_{11} = 0, & -x_5 - x_{15} = 0, & -x_9 + x_{14} = 0, \\ \boxed{-x_4 - x_7 = 0}\,, & -x_1 - x_{11} = 0, & x_2 - x_{15} = 0, \\ -x_5 + x_{12} = 0, & x_6 + x_{16} = 0, & \boxed{x_{10} + x_{13} = 0}\,. \end{array}$$

Those highlighted equations read

$$x_4 = x_{10}, \quad -x_7 = -x_{13}, \quad x_4 = -x_7, \quad x_{10} = -x_{13}.$$

These yield

$$x_4 = -x_7 = x_{10} = -x_{13}.$$

In other words, using a parameter d,

$$\Big(x_4, \; x_7, \; x_{10}, \; x_{13} \Big) = \Big(d, \; -d, \; d, \; -d \Big).$$

$\Big[$ **Answer** $\Big]$ **:**

$$\left(\begin{array}{llll} x_1, & x_2, & x_3, & x_4, \\ x_5, & x_6, & x_7, & x_8 \\ x_9, & x_{10}, & x_{11}, & x_{12}, \\ x_{13}, & x_{14}, & x_{15}, & x_{16} \end{array} \right) = \left(\begin{array}{llll} \underline{\;0\;}, & \underline{\;b\;}, & \underline{\;c\;}, & \underline{\;d\;}, \\ \underline{-b}, & \underline{\;0\;}, & \underline{-d}, & \underline{\;c\;}, \\ \underline{-c}, & \underline{\;d\;}, & \underline{\;0\;}, & \underline{-b}, \\ \underline{-d}, & \underline{-c}, & \underline{\;b\;}, & \underline{\;0\;} \end{array} \right).$$

(2) Substitute the parametrized solution in (1) into

$$(*) \quad \begin{array}{rcl@{\qquad}rcl} x_1x_6 - x_2x_5 + x_3x_8 - x_4x_7 &=& 1, & x_9x_{14} - x_{10}x_{13} + x_{11}x_{16} - x_{12}x_{15} &=& 1, \\ x_1x_{10} - x_2x_9 + x_3x_{12} - x_4x_{11} &=& 0, & x_5x_{14} - x_6x_{13} + x_7x_{16} - x_8x_{15} &=& 0, \\ x_1x_{14} - x_2x_{13} + x_3x_{16} - x_4x_{15} &=& 0, & x_5x_{10} - x_6x_9 + x_7x_{12} - x_8x_{11} &=& 0, \\ x_1x_8 - x_4x_5 + x_2x_7 - x_3x_6 &=& 0, & x_9x_{16} - x_{12}x_{13} + x_{10}x_{15} - x_{11}x_{14} &=& 0, \\ x_1x_{12} - x_4x_9 + x_2x_{11} - x_3x_{10} &=& 0, & x_5x_{16} - x_8x_{13} + x_6x_{15} - x_7x_{14} &=& 0, \\ x_1x_{16} - x_4x_{13} + x_2x_{15} - x_3x_{14} &=& 1, & x_5x_{12} - x_8x_9 + x_6x_{11} - x_7x_{10} &=& 1, \\ x_1x_7 - x_3x_5 - x_2x_8 + x_4x_6 &=& 0, & x_9x_{15} - x_{11}x_{13} - x_{10}x_{16} + x_{12}x_{14} &=& 0, \\ x_1x_{11} - x_3x_9 - x_2x_{12} + x_4x_{10} &=& 1, & x_5x_{15} - x_7x_{13} - x_6x_{16} + x_8x_{14} &=& -1, \\ x_1x_{15} - x_3x_{13} - x_2x_{16} + x_4x_{14} &=& 0, & x_5x_{11} - x_7x_9 - x_6x_{12} + x_8x_{10} &=& 0. \end{array}$$

The result is

$$
\begin{aligned}
0\cdot 0 - b(-b) + cc - d(-d) &= 1, & (-c)(-c) - d(-d) + 0\cdot 0 - (-b)b &= 1,\\
0d - b(-c) + c(-b) - d0 &= 0, & (-b)(-c) - 0(-d) + (-d)0 - cb &= 0,\\
0(-c) - b(-d) + c0 - db &= 0, & (-b)d - 0(-c) + (-d)(-b) - c0 &= 0,\\
0c - d(-b) + b(-d) - c0 &= 0, & (-c)0 - (-b)(-d) + db - 0(-c) &= 0,\\
0(-b) - d(-c) + b0 - cd &= 0, & (-b)0 - c(-d) + 0b - (-d)(-c) &= 0,\\
0\cdot 0 - d(-d) + bb - c(-c) &= 1, & (-b)(-b) - c(-c) + 0\cdot 0 - (-d)d &= 1,\\
0(-d) - c(-b) - bc + d0 &= 0, & (-c)b - 0(-d) - d0 + (-b)(-c) &= 0,\\
0\cdot 0 - c(-c) - b(-b) + dd &= 1, & (-b)b - (-d)(-d) - 0\cdot 0 + c(-c) &= -1,\\
0b - c(-d) - b0 + d(-c) &= 0, & (-b)0 - (-d)(-c) - 0(-b) + cd &= 0.
\end{aligned}
$$

These are simplified as

$$
\begin{aligned}
b^2 + c^2 + d^2 &= 1, & b^2 + c^2 + d^2 &= 1,\\
0 &= 0, & 0 &= 0,\\
0 &= 0, & 0 &= 0,\\
0 &= 0, & 0 &= 0,\\
0 &= 0, & 0 &= 0,\\
b^2 + c^2 + d^2 &= 1, & b^2 + c^2 + d^2 &= 1,\\
0 &= 0, & 0 &= 0,\\
b^2 + c^2 + d^2 &= 1, & -b^2 - c^2 - d^2 &= -1,\\
0 &= 0, & 0 &= 0.
\end{aligned}
$$

[**Answer**] : $b^2 + c^2 + d^2 = 1.$

(3) Let's move back to the substitution done in (2):

$$
\begin{aligned}
0\cdot 0 - b(-b) + cc - d(-d) &= 1, & (-c)(-c) - d(-d) + 0\cdot 0 - (-b)b &= 1,\\
0d - b(-c) + c(-b) - d0 &= 0, & (-b)(-c) - 0(-d) + (-d)0 - cb &= 0,\\
0(-c) - b(-d) + c0 - db &= 0, & (-b)d - 0(-c) + (-d)(-b) - c0 &= 0,\\
0c - d(-b) + b(-d) - c0 &= 0, & (-c)0 - (-b)(-d) + db - 0(-c) &= 0,\\
0(-b) - d(-c) + b0 - cd &= 0, & (-b)0 - c(-d) + 0b - (-d)(-c) &= 0,\\
0\cdot 0 - d(-d) + bb - c(-c) &= 1, & (-b)(-b) - c(-c) + 0\cdot 0 - (-d)d &= 1,\\
0(-d) - c(-b) - bc + d0 &= 0, & (-c)b - 0(-d) - d0 + (-b)(-c) &= 0,\\
0\cdot 0 - c(-c) - b(-b) + dd &= 1, & (-b)b - (-d)(-d) - 0\cdot 0 + c(-c) &= -1,\\
0b - c(-d) - b0 + d(-c) &= 0, & (-b)0 - (-d)(-c) - 0(-b) + cd &= 0.
\end{aligned}
$$

In these, if you replace 0s <u>on the left-hand side</u> with as, then accordingly you obtain

$$
\begin{aligned}
aa - b(-b) + cc - d(-d) &= 1, & (-c)(-c) - d(-d) + aa - (-b)b &= 1,\\
ad - b(-c) + c(-b) - da &= 0, & (-b)(-c) - a(-d) + (-d)a - cb &= 0,\\
a(-c) - b(-d) + ca - db &= 0, & (-b)d - a(-c) + (-d)(-b) - ca &= 0,\\
ac - d(-b) + b(-d) - ca &= 0, & (-c)a - (-b)(-d) + db - a(-c) &= 0,\\
a(-b) - d(-c) + ba - cd &= 0, & (-b)a - c(-d) + ab - (-d)(-c) &= 0,\\
aa - d(-d) + bb - c(-c) &= 1, & (-b)(-b) - c(-c) + aa - (-d)d &= 1,\\
a(-d) - c(-b) - bc + da &= 0, & (-c)b - a(-d) - da + (-b)(-c) &= 0,\\
aa - c(-c) - b(-b) + dd &= 1, & (-b)b - (-d)(-d) - aa + c(-c) &= -1,\\
ab - c(-d) - ba + d(-c) &= 0, & (-b)a - (-d)(-c) - a(-b) + cd &= 0.
\end{aligned}
$$

These are simplified as

$$
\begin{aligned}
a^2 + b^2 + c^2 + d^2 &= 1, & a^2 + b^2 + c^2 + d^2 &= 1,\\
0 &= 0, & 0 &= 0,\\
0 &= 0, & 0 &= 0,\\
0 &= 0, & 0 &= 0,\\
0 &= 0, & 0 &= 0,\\
a^2 + b^2 + c^2 + d^2 &= 1, & a^2 + b^2 + c^2 + d^2 &= 1,\\
0 &= 0, & 0 &= 0,\\
a^2 + b^2 + c^2 + d^2 &= 1, & -a^2 - b^2 - c^2 - d^2 &= -1,\\
0 &= 0, & 0 &= 0.
\end{aligned}
$$

So the following answer works for the system $(*)$:

[**Answer**] **:**

$$
\begin{pmatrix} x_1, & x_2, & x_3, & x_4, \\ x_5, & x_6, & x_7, & x_8 \\ x_9, & x_{1a}, & x_{11}, & x_{12}, \\ x_{13}, & x_{14}, & x_{15}, & x_{16} \end{pmatrix} = \begin{pmatrix} \underline{a}, & \underline{b}, & \underline{c}, & \underline{d}, \\ \underline{-b}, & \underline{a}, & \underline{-d}, & \underline{c}, \\ \underline{-c}, & \underline{d}, & \underline{a}, & \underline{-b}, \\ \underline{-d}, & \underline{-c}, & \underline{b}, & \underline{a} \end{pmatrix},
$$

with the constraint $\boxed{a^2 + b^2 + c^2 + d^2 = 1}$.

$\star$ The above argument only proves that this last answer is a part of the solution set for the system $(*)$.

$\star$ The issue of whether the above answer in (3) exhausts the entire solution set for the system $(*)$, and also where the system $(*)$ comes from, how it is related to the system $(**)$, *etc.* will be addressed in the next extra assignment $\big($"Extra Credit Homework – II"$\big)$.