Math 290 ELEMENTARY LINEAR ALGEBRA

SOLUTION FOR PRACTICE EXAM – I (09/27)

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[I] (20pts) (1)

$$I = \begin{bmatrix} \boxed{1} & \boxed{0} \\ \boxed{0} & \boxed{1} \end{bmatrix}.$$

(2) The system

$$\begin{cases} 3x - 2y = 6, \\ 5x + y = -4 \end{cases}$$

is rewritten as

$$\underbrace{\begin{bmatrix} \boxed{3} & \boxed{-2} \\ \boxed{5} & \boxed{1} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \boxed{x} \\ \boxed{y} \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} \boxed{6} \\ \boxed{-4} \end{bmatrix}}_{\boldsymbol{b}}.$$

(3) Solve the system in (2):

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = \overbrace{\frac{1}{\boxed{3 \cdot 1 - (-2) \cdot 5}} \begin{bmatrix} \boxed{1} & \boxed{2} \\ \boxed{-5} & \boxed{3} \end{bmatrix}}^{A^{-1}} \overbrace{\begin{bmatrix} \boxed{6} \\ \boxed{-4} \end{bmatrix}}^{\boldsymbol{b}}$$

$$= \frac{1}{\boxed{13}} \begin{bmatrix} \boxed{-2} \\ \boxed{-42} \end{bmatrix}.$$

[II] (20pts) Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix}$.

(1a)
$$\det A = \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} = 1 \cdot 2 - 2 \cdot \left(-1 \right)$$
$$= 4.$$

(1b)
$$\det B = \begin{vmatrix} 3 & 4 \\ 1 & 3 \end{vmatrix} = 3 \cdot 3 - 4 \cdot 1$$
$$= 5.$$

(2)
$$\det \left(AB \right) = \det \begin{bmatrix} 5 & 10 \\ -1 & 2 \end{bmatrix} = 5 \cdot 2 - 10 \cdot \left(-1 \right)$$
$$= 20.$$

(3) True or false : $\det \left(AB \right) = \left(\det A \right) \left(\det B \right)$.

The answer is ‘True’.

[III] (30pts) (1) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(1a) True or false : $IA = A$.

— The answer is ‘True’.

(1b) True or false : $AI = A$.

— The answer is ‘True’.

(2) For A as in (1), suppose $ad - bc \neq 0$.

(2a) True or false : $AA^{-1} = I$.

— The answer is 'True'.

(2b) True or false : $A^{-1}A = I$.

— The answer is 'True'.

(3) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$. Suppose $AB = I$.

(3a) True or false : $BA = I$.

— The answer is 'True'.

(3b) True or false : $B = A^{-1}$.

— The answer is 'True'.

(3c) True or false : $A = B^{-1}$.

— The answer is 'True'.

(3d) Explain why you chose the answers you chose for (3a), (3b) and (3c) above.

Reason: If $AB = I$, then by the product rule, $\det A \neq 0$.

So A^{-1} exists. Multiply A^{-1} from the left to the two sides

of $AB = I$ and get $B = A^{-1}$, thanks to the associativity law.

By (2b), $BA = I$. Hence also $A = B^{-1}$.

[IV] (10pts) <u>True or False</u> :

For

$$A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \quad B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}, \quad C = \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix}, \quad D = \begin{bmatrix} a_4 & b_4 \\ c_4 & d_4 \end{bmatrix},$$

(1a) $$A\Big(B+C\Big) = AB + AC.$$

— The answer is 'True'.

(1b) $$\Big(B+C\Big)D = BD + CD.$$

— The answer is 'True'.

[V] (10pts) The determinant of

$$A = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} :$$

$$\det A = 0 \cdot \begin{vmatrix} 0 & -a \\ a & 0 \end{vmatrix} - \Big(-c\Big) \cdot \begin{vmatrix} c & -a \\ -b & 0 \end{vmatrix} + b \cdot \begin{vmatrix} c & 0 \\ -b & a \end{vmatrix}$$

$$= 0 + c \cdot \Big(c \cdot 0 - (-a)(-b)\Big) + b \cdot \Big(c \cdot a - 0 \cdot (-b)\Big)$$

$$= 0 + c \cdot \Big(-ab\Big) + b \cdot ca$$

$$= -abc + abc$$

$$= 0.$$

[VI] (20pts) (1) Circle all matrices that are in reduced row echelon form:

(1a) $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. (1b) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

(1c) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. (1d) $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$.

(1e) $\begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. (1f) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

[**Answer**]: (1c), (1e) and (1f) are in reduced row echelon form. Others are not in reduced row echelon form.

(2)

$$\begin{bmatrix} \boxed{1} & \boxed{*} & \boxed{0} & \boxed{*} & \boxed{*} & \boxed{0} \\ \boxed{0} & \boxed{0} & \boxed{1} & \boxed{*} & \boxed{*} & \boxed{0} \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{1} \end{bmatrix}.$$

[VII] (10pts) Let

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad B = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix},$$

$$C = \begin{bmatrix} \cos\left(\theta + \phi\right) & -\sin\left(\theta + \phi\right) \\ \sin\left(\theta + \phi\right) & \cos\left(\theta + \phi\right) \end{bmatrix}.$$

(1) AB

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{\cos\theta} \cdot \boxed{\cos\phi} - \boxed{\sin\theta} \cdot \boxed{\sin\phi} & -\left(\boxed{\cos\theta} \cdot \boxed{\sin\phi} + \boxed{\sin\theta} \cdot \boxed{\cos\phi}\right) \\ \boxed{\cos\theta} \cdot \boxed{\sin\phi} + \boxed{\sin\theta} \cdot \boxed{\cos\phi} & \boxed{\cos\theta} \cdot \boxed{\cos\phi} - \boxed{\sin\theta} \cdot \boxed{\sin\phi} \end{bmatrix}.$$

(2) Know $C = AB$. Write out its consequence:

$$\cos\left(\theta + \phi\right) = \boxed{\cos\theta} \cdot \boxed{\cos\phi} - \boxed{\sin\theta} \cdot \boxed{\sin\phi},$$

$$\sin\left(\theta + \phi\right) = \boxed{\cos\theta} \cdot \boxed{\sin\phi} + \boxed{\sin\theta} \cdot \boxed{\cos\phi}.$$

⋆ Acknowledge that the two equations in (2) are the addition formulas for trigs.

[VIII] (10pts) Find the <u>characteristic polynomial</u>, and the <u>eigenvalues</u>, of

$$A = \begin{bmatrix} 6 & 1 \\ 2 & 7 \end{bmatrix}.$$

Solution.

$$\begin{aligned} \chi_A(\lambda) &= \det\left(\lambda I - A\right) \\ &= \begin{vmatrix} \lambda - 6 & -1 \\ -2 & \lambda - 7 \end{vmatrix} \\ &= \left(\lambda - 6\right)\left(\lambda - 7\right) - \left(-1\right) \cdot \left(-2\right) \\ &= \lambda^2 - 13\lambda + 40. \end{aligned}$$

This is factored as

$$\left(\lambda - 5\right)\left(\lambda - 8\right).$$

So, the eigenvalues of A are

$$\lambda = 5 \qquad \text{and} \qquad \lambda = 8.$$