Math 290 ELEMENTARY LINEAR ALGEBRA

REVIEW OF LECTURES – X

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§10. Matrices in arbitrary size.

A quick run down:

(0) Brute-force attempt to solve systems of linear equations quickly gives way to building up a matrix theory. Determinants lie at the heart of it. Only with a firm grasp of matrices and determinants will we have a shot at getting a hold of systems of linear equations. We will be faithful to this scope for the entire semester.

(1) We defined the 2×2, and the 3×3, determinants. The 3×3 determinant turns out to be made of 2×2 determinants (‘co-factor expansion’).

(2) (just a heads-up) Co-factor expansions exist for larger size determinants, namely: the 4×4 determinant is made of 3×3 determinants — more generally, the $n \times n$ determinant is made of $(n-1) \times (n-1)$ determinants. Here, the definition of the $n \times n$ determinant for $n \geq 4$ pending.

(3) We have defined multiplication AB of two 2×2 matrices A, B. We have proved $\det(AB) = (\det A)(\det B)$ (the product formula). It illustrates that the notion of determinants and the notion of matrix multiplication totally go together.

(4) We have defined multiplication AB of two 3×3 matrices A, B. The same for larger size matrices still pending.

(5) (also a heads-up) The same product formula holds true when A and B are both $n \times n$. Despite its innocuous looking when written in one line, the formula is non-trivial: The “spelt-out version” of it for $n = 4$ already takes up an entire page (“Review of Lectures – VI”, page 8). The number of terms that show up on each side of the formula (after expanding all the parentheses) grows exponentially as n grows. Before long it goes above the computer’s capacity. Proving it in an *ex machina* way is feasible within the scope of Math 290. We shoot for it. More generally, you are going to learn things which your computer cannot replace. In math, *theories* supersede mindless calculations (mathematicians’ job is to discover such theories).

(6) Matrix arithmetic, mainly for 2×2 case.

- Inverse A^{-1} of a matrix A (so far 2×2 and 3×3).
- The identity matrix I (so far 2×2 and 3×3).
- Parallelism between the role of the matrix I and the role of the number 1.
- Matrix addition and subtraction (so far 2×2).
- Associative and distributive laws. Powers A^k (so far 2×2).

(7) (just a sneak preview) Eigenvalues and the characteristic polynomial of a matrix A (so far 2×2). This topic is going to be one of the highlights of the second-half of the semester.

(8) Gaussian elimination method. How to frame it in the language of matrices. Reduced row echelon form. How this has a bearing on the inverse of a matrix.

• That's what we have covered so far. We want to push this direction further, as the Break (the end of the first-half) is sneaking up on us as we speak. With that in mind, I want to announce one thing: I am going to change the narrative somewhat. Thus far our approach has been *heuristic*: When I introduce something, I always stop and tell you why we do this, why we do that, offering the *raison d'être* of every concept, that this is not some useless junk, but it is relevant for such and such reason. All that despite the fact that once armed with a deeper knowledge you get to see all of that (which is going to happen before the semester ends, by design) so it will ultimately become redundant. In the future, I will gear more towards helping you acclimate to the style how mathematicians articulate things. In any professional mathematical texts (that includes math textbooks for graduate courses), the author follows a well-established writing style, consistent with the standards of rigor in research math. It is more-or-less as follows:

$$\text{Definition(s)} \implies \text{Theorem(s) (and formula(s))} \implies \text{Proof(s)}$$
$$\implies \text{Another definition(s)} \implies \text{Theorem(s) (and formula(s))} \implies \text{Proof(s)}$$
$$\implies \text{Occasionally examples.}$$

I'm going to incorporate this very format, to a *moderate* extent. The reason why I do that is firstly, it is *efficient*. In the first nine lectures, I have sacrificed efficiency. I did that deliberately. As a side-track, we professional mathematicians too need to hear some 'spiel' when we attend someone's (our peers') professional seminars and colloquium lectures, which are typically about new theories the speaker has just unleashed.

And 'spiel' is the part we get the most out of it — remember, hundred different mathematicians hold hundred different areas of expertise. The speaker will then go on to disclose how everything falls into place so we, the audience, can trace it if we care enough — which we seldom do due to the slim odds a connection exists between the talk and our own research. But we go out and do it when our gut says so, and it sometimes pays off if you are lucky, namely, it leads to a new discovery. That's one time-tested way to engineer a new theory. Then you return a favor to the mathematicians' community by giving a presentation yourself. Within our professional circle, the word 'math' means this very process, a mass undertaking, a succession of submissions of new math theories by experts, a collective endeavor to upgrade human knowledge in cutting-edge math.

Let me dwell on this peer presentation thing just a tiny bit more (bear with me) beause it's finally relevant to what I was saying earlier about this class: All things considered, the most effective approach, if you are the speaker, is to assume that the audience is "*infinitely ignorant, and infinitely intelligent*". As a lecture-giver, you must fill in the "infinite ignorance" part of the audience. I know that's a '*platonic ideal*', but trust me, it is viable for undergraduate classes, like this one (Math 290), save that you are not infinitely ignorant: You are already equipped with some firm background knowledge on the prerequisite math. So, where am I getting at? Yes, I *will* assume (as I always have) you guys are infinitely intelligent too. The allotted number of hours is limited, and I want to share a chunk of knowledge. So I have to strike a balance between '*spiel*' and '*efficiency*'. But no worries, it's not like this class suddenly becomes 'esoteric' (← meaning 'as clear as mud'). Let me put it nicely as follows: 'The slope we are climbing gets steeper from now on'. So, here we go, let me throw one definition, where the narrative is very characteristic of the aforementioned "mathematicians' writing style":

Definition (a matrix). A <u>matrix</u> is

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

where a_{ij} are <u>scalars</u> (= real numbers). Each individual scalar a_{ij} is called an <u>entry</u> of the matrix A. When we need to emphasize the location of a_{ij}, we say that a_{ij} is the (i, j)-th entry of the matrix A, the i-th from the top and the j-th from the left.

• In the above, the number of <u>rows</u> is m and the number of <u>columns</u> is n. We then call the matrix to have <u>size $m \times n$</u>.

• The size description should always be

$$\boxed{\Big(\text{ the number of rows }\Big) \times \Big(\text{ the number of columns }\Big)} \; .$$

• Always, the i-th row means the i-th row <u>from the top</u>, and the j-th column means the j-th column <u>from the left</u>.

• It is common to omit the bracket $[\quad]$ on a <u>1×1</u> matrix. We may write

$$a, \qquad \text{instead of} \qquad [a].$$

— How was that? You might have felt that my sound and tone got *drier*, if ever so slightly. I'm going to maintain this style. So I want you to hang on every word I say. Trust me, you'll get used to this quickly.

Example 1. We may write a general 2×2 matrix as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We may write a general 3×3 matrix as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}.$$

We may write a general 2×4 matrix as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} a & b & c & d \\ p & q & r & s \end{bmatrix}.$$

Exercise 1. Write out each of the following:

(1) The general 4×4 matrix. (2) The general 2×5 matrix.

(3) The general 3×6 matrix.

- **Identification of two matrices.**

  Consider two matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & a_{12} & \cdots & a_{1\ell} \\ b_{21} & a_{22} & \cdots & a_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & a_{k2} & \cdots & a_{k\ell} \end{bmatrix}.$$

As you can see,

$$A \text{ is in size } m \times n, \qquad \text{whereas} \qquad B \text{ is in size } k \times \ell.$$

Now, A and B are **equal**, when their size descriptions match, namely, $m = k$, and $n = \ell$, and moreover their corresponding entries match, namely,

$$\boxed{a_{ij} = b_{ij}} \quad \text{holds for each } i,\ j\ .$$

In this situation, we write

$$\boxed{A = B}\ .$$

If A and B are **not** equal, then we write

$$\boxed{A \neq B}\ .$$

Example 2.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

— So far so good?

- **Row vectors, Column vectors.**

  A matrix of size $1 \times n$ is called a row vector (of length n). A matrix of size $m \times 1$ is called a column vector (of length m).

Example 3. A (general) row vector of length 2 is $\begin{bmatrix} a & b \end{bmatrix}$, or $\begin{bmatrix} a_1 & a_2 \end{bmatrix}$.

A (general) column vector of length 2 is $\begin{bmatrix} a \\ b \end{bmatrix}$, or $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$.

A (general) row vector of length 3 is $\begin{bmatrix} a & b & c \end{bmatrix}$, or $\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$.

A (general) column vector of length 3 is $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, or $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$.

Exercise 2. Write out each of the following:

(1) A general row vector of length 4.

(2) A general column vector of length 4.

- **Matrix addition.**

For two matrices A and B which are in the same size, their sum $A + B$ is defined. $A + B$ is a matrix having the same size as A and B.

Definition. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}.$$

Note that A and B are both in size $m \times n$. We define

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}.$$

- We may paraphrase the definition as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}.$$

In short, the matrix addition is defined as an "entrywise" addition .

- If A and B are **not** in the same size, then $A+B$ is **undefined** .

Example 4. (1) For $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & -2 \\ 4 & 2 \end{bmatrix}$, we have

$$A + B = \begin{bmatrix} 1 + (-3) & 2 + (-2) \\ 2 + 4 & 1 + 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 6 & 3 \end{bmatrix}.$$

(2) For

$$C = \begin{bmatrix} 0 & 3 & -1 & 2 \\ -2 & 4 & -6 & 3 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 2 & 0 & 1 \\ 1 & -3 & 1 & 0 \\ 0 & 1 & 2 & 5 \end{bmatrix},$$

we have

$$C + D = \begin{bmatrix} 0+2 & 3+2 & -1+0 & 2+1 \\ -2+1 & 4+(-3) & -6+1 & 3+0 \\ 1+0 & 1+1 & 1+2 & 0+5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 & -1 & 3 \\ -1 & 1 & -5 & 3 \\ 1 & 2 & 3 & 5 \end{bmatrix}.$$

(3) For $\boldsymbol{u} = \begin{bmatrix} 3 & 3 & 2 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} -1 & -2 & 0 \end{bmatrix}$, we have

$$\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} 3 + (-1) & 3 + (-2) & 2 + 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \end{bmatrix}.$$

(4) For $E = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & 2 \end{bmatrix}$, $F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $E + F$ is undefined.

(5) For $G = \begin{bmatrix} 0 & 4 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$, $H = \begin{bmatrix} 0 & 8 & 6 \\ -1 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, $G + H$ is undefined.

Exercise 3. Calculate, if feasible.

(1) $A + B$, where $A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}$, $B = \begin{bmatrix} -2 & -4 & -6 & -8 \\ -1 & -3 & -5 & -7 \end{bmatrix}$.

(2) $\begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

(3) $\boldsymbol{u} + \boldsymbol{v}$, where $\boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} 1 \\ 3 \\ 9 \\ 27 \end{bmatrix}$.

(4) $\begin{bmatrix} 1 & 3 & 6 & 10 & 15 \end{bmatrix} + \begin{bmatrix} 3 & 6 & 10 & 15 & 21 \end{bmatrix}$.

(5) $A + B$, where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(6) $\begin{bmatrix} 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$.

(7) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$.

- **Scalar multiplication.** For a matrix A and a scalar s, sA is defined.

Definition. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Let s be a scalar. We define

$$s\,A = \begin{bmatrix} s\,a_{11} & s\,a_{12} & \cdots & s\,a_{1n} \\ s\,a_{21} & s\,a_{22} & \cdots & s\,a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s\,a_{m1} & s\,a_{m2} & \cdots & s\,a_{mn} \end{bmatrix}.$$

- We may paraphrase the definition as

$$s \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} s\,a_{11} & s\,a_{12} & \cdots & s\,a_{1n} \\ s\,a_{21} & s\,a_{22} & \cdots & s\,a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s\,a_{m1} & s\,a_{m2} & \cdots & s\,a_{mn} \end{bmatrix}.$$

In short, <u>a scalar multiplication of a matrix is defined as a multiplication of "the same scalar to each entry"</u>.

- In particular, we may define $\big(-1\big)\,A$ for a matrix A. We often write it as $-A$:

$$\boxed{\Big(-1\Big)A = -A}.$$

That is, for $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, we have

$$-A = \begin{bmatrix} -a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & -a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{m1} & -a_{m2} & \cdots & -a_{mn} \end{bmatrix}.$$

Exercise 4. Calculate.

(1) $2A$, where $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$

(2) $-4A$, where $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \\ 2 & 4 \end{bmatrix}.$

(3) $-A$, where $A = \begin{bmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{bmatrix}.$

(4) $10\boldsymbol{u}$, where $\boldsymbol{u} = \begin{bmatrix} 12 & 12 & 12 & 12 \end{bmatrix}.$

- **A linear combination of matrices.**

For two matrices A and B which are in the same size, and for two scalars s and t, the linear combination

$$\boxed{sA + tB}$$

is just the sum of sA and tB.

- As a special case, we may consider $1 \cdot A + (-1) \cdot B$. We may write it as $A - B$:

$$\boxed{A - B = 1 \cdot A + \left(-1\right) \cdot B}\ .$$

- It makes sense to more generally consider the linear combination

$$s_1 A_1 + s_2 A_2 + \cdots + s_r A_r$$

for matrices $A_1,\ A_2,\ \cdots,\ A_r$ in the same size, and for scalars $s_1,\ s_2,\ \cdots,\ s_r$.

Example 5. For

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 4 \end{bmatrix}, \qquad \text{and} \qquad B = \begin{bmatrix} 2 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix},$$

we have

$$2\,A = \begin{bmatrix} 2\cdot 2 & 2\cdot 1 & 2\cdot 1 \\ 2\cdot(-1) & 2\cdot(-1) & 2\cdot 4 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ -2 & -2 & 8 \end{bmatrix},$$

$$-\,B = \begin{bmatrix} -2 & 3 & -4 \\ 3 & -1 & 2 \end{bmatrix}, \qquad \text{and}$$

$$2\,A - B = (2\,A) + (-\,B) = \begin{bmatrix} 4 & 2 & 2 \\ -2 & -2 & 4 \end{bmatrix} + \begin{bmatrix} -2 & 3 & -4 \\ 3 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 & -2 \\ 1 & -3 & 6 \end{bmatrix}.$$

Exercise 5. (1) For $A = \begin{bmatrix} 6 & -1 \\ 2 & 4 \\ -3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 \\ -1 & 5 \\ 1 & 10 \end{bmatrix}$, find $A - B$.

(2) For $A = \begin{bmatrix} 4 & 1 & -6 \\ 0 & 3 & 2 \\ -3 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 5 \\ -4 & 6 & 1 \\ -3 & 4 & 6 \end{bmatrix}$, find $5\,A + 2\,B$.

(3) For $\boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$, $\boldsymbol{w} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, find $3\boldsymbol{u} + 4\boldsymbol{v} + 5\boldsymbol{w}$.

• **Matrix multiplication.**

For two matrices A and B, their product AB is defined, whenever the number of columns of A equals the number of rows of B. AB is a matrix whose number of rows equals that of A, and whose number of columns equals that of B.

Definition. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix}, \qquad \text{and} \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rn} \end{bmatrix}.$$

Note that r is the number of columns for A, as well as the number of rows for B at the same time. So

$$\boxed{\text{the number of columns for } A = \text{the number of rows for } B.}$$

Then we define their product AB as follows:

$$AB = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

is a matrix having size $m \times n$, and having entries c_{ij} which are decided by the following rule:

Rule. To find the $(i,\ j)$-th entry c_{ij} of the product AB, single out the row i from the matrix A, and the column j from the matrix B:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ \vdots & \vdots & & \vdots \\ \boxed{a_{i1} \quad a_{i2} \quad \cdots \quad a_{ir}} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix}, \qquad \text{and} \qquad B = \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{1n} \\ \vdots & & \vdots & & \vdots \\ b_{r1} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix}.$$

Call them $\boldsymbol{a}_i$ and $\boldsymbol{b}_j$, respectively:

$$\boldsymbol{a}_i = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{ir} \end{bmatrix}, \qquad \text{and} \qquad \boldsymbol{b}_j = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{rj} \end{bmatrix}.$$

Multiply the corresponding entries for the row $\boldsymbol{a}_i$ and the column $\boldsymbol{b}_j$ together, then add up the resulting products, and that's $\boxed{c_{ij}}$:

$$\boxed{\quad a_{i1}\,b_{1j} \;+\; a_{i2}b_{2j} \;+\; \cdots \;+\; a_{ir}b_{rj} \;=\; c_{ij}. \quad}$$

This $\boxed{c_{ij}}$ will sit at the $\big(i,\ j\big)$-th entry of the product AB.

- Consider the special case when B is a column vector $\boldsymbol{b}$:

$$A \;=\; \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix}, \qquad \text{and} \qquad \boldsymbol{b} \;=\; \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \end{bmatrix}.$$

Then

$$A\boldsymbol{b} \;=\; \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \end{bmatrix} \;=\; \begin{bmatrix} a_{11}b_1 \;+\; a_{12}b_2 \;+\; \cdots \;+\; a_{1r}b_r \\ a_{21}b_1 \;+\; a_{22}b_2 \;+\; \cdots \;+\; a_{2r}b_r \\ \vdots \\ a_{m1}b_1 \;+\; a_{m2}b_2 \;+\; \cdots \;+\; a_{mr}b_r \end{bmatrix}.$$

- **Note.** The same can also be written as

$$A\boldsymbol{b} \;=\; b_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \;+\; b_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} \;+\; \cdots \;+\; b_r \begin{bmatrix} a_{1r} \\ a_{2r} \\ \vdots \\ a_{mr} \end{bmatrix}.$$

- Similarly, consider the special case when A is a row vector $\boldsymbol{a}$:

$$\boldsymbol{a} \;=\; \begin{bmatrix} a_1 & a_2 & \cdots & a_r \end{bmatrix}, \qquad \text{and} \qquad B \;=\; \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rn} \end{bmatrix}.$$

Then

$$\boldsymbol{a}\,B \;=\; \begin{bmatrix} a_1 & a_2 & \cdots & a_r \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rn} \end{bmatrix}$$

$$= \Big[a_1\,b_{11} + a_2\,b_{21} + \cdots + a_r\,b_{r1} \qquad a_1\,b_{12} + a_2\,b_{22} + \cdots + a_r\,b_{r2}$$

$$\cdots \quad a_1\,b_{1n} + a_2\,b_{2n} + \cdots + a_r\,b_{rn} \Big].$$

- **Note.** The same can also be written as

$$\begin{aligned} \boldsymbol{a}\,B \;=\; & \quad a_1 \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \end{bmatrix} \\ & + \; a_2 \begin{bmatrix} b_{21} & b_{22} & \cdots & b_{2n} \end{bmatrix} \\ & + \qquad \cdots \qquad \cdots \\ & + \; a_r \begin{bmatrix} b_{r1} & b_{r2} & \cdots & b_{rn} \end{bmatrix}. \end{aligned}$$

- As we have already seen in the case A and B are both 2×2, (and also the case A and B are both 3×3), $\underline{AB \text{ and } BA \text{ need not be equal}}$, even if both AB and BA are defined and are in the same size.

Example 6. For $A = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$, we have

$$\begin{aligned} A\,B &= \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \\ &= 3 \cdot 2 + 2 \cdot 3 + 1 \cdot 0 \\ &= 12, \end{aligned}$$

whereas

$$BA = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cdot 3 & 2 \cdot 2 & 2 \cdot 1 \\ 3 \cdot 3 & 3 \cdot 2 & 3 \cdot 1 \\ 0 \cdot 3 & 0 \cdot 2 & 0 \cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 4 & 2 \\ 9 & 6 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

- As Example 6 shows, <u>AB and BA need not have the same size</u>, even if both AB and BA are defined.

Example 7. For $A = \begin{bmatrix} -1 & 3 \\ 4 & -5 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 7 \end{bmatrix}$, we have

$$AB = \begin{bmatrix} -1 & 3 \\ 4 & -5 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} (-1) \cdot 1 + 3 \cdot 0 & (-1) \cdot 2 + 3 \cdot 7 \\ 4 \cdot 1 + (-5) \cdot 0 & 4 \cdot 2 + (-5) \cdot 7 \\ 0 \cdot 1 + 2 \cdot 0 & 0 \cdot 2 + 2 \cdot 7 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 19 \\ 4 & -27 \\ 0 & 14 \end{bmatrix},$$

BA is <u>undefined</u>.

- As Example 7 shows, <u>BA needs not be defined</u>, even if AB is defined.

Exercise 6. (1) For

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{-1}{8} & 0 \\ 0 & 0 & \frac{1}{7} \end{bmatrix},$$

find AB and BA. If undefined, write undefined.

(2) For

$$A = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 \end{bmatrix},$$

find AB and BA. If undefined, write undefined.

(3) For

$$A = \begin{bmatrix} 4 & 1 & 2 & -3 \\ -1 & 4 & 3 & 2 \\ -2 & -3 & 4 & -1 \\ 3 & -2 & 1 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} -4 & 1 & 2 & -3 \\ -1 & -4 & 3 & 2 \\ -2 & -3 & -4 & -1 \\ 3 & -2 & 1 & -4 \end{bmatrix},$$

find AB and BA. If undefined, write undefined.

- **A matrix partition into columns/rows.**

  A matrix can be "partitioned" into its columns. If B is a matrix

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rn} \end{bmatrix},$$

then we may give each column a name

$$\boldsymbol{b}_1 = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{r1} \end{bmatrix}, \qquad \boldsymbol{b}_2 = \begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{r2} \end{bmatrix}, \qquad \cdots \qquad \boldsymbol{b}_n = \begin{bmatrix} b_{1n} \\ b_{2n} \\ \vdots \\ b_{rn} \end{bmatrix},$$

and think of B as $\begin{bmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_n \end{bmatrix}$. We call $\boldsymbol{b}_j$ the j-th column vector of B.

Formula 1. Let $B = \begin{bmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_n \end{bmatrix}$ be as above. Let A be any matrix whose number of columns is r. Then

$$AB = \begin{bmatrix} A\boldsymbol{b}_1 & A\boldsymbol{b}_2 & \cdots & A\boldsymbol{b}_n \end{bmatrix}.$$

Example 8. $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}.$$

The first column of the resulting matrix is $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix}$.

The second column of the resulting matrix is $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix}$.

- Similarly, a matrix can be "partitioned" into its rows. If A is a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix},$$

then we may give each column a name

$$\begin{aligned} \boldsymbol{a}_1 &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \end{bmatrix}, \\ \boldsymbol{a}_2 &= \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2r} \end{bmatrix}, \\ \vdots & \qquad \vdots \qquad \vdots \qquad\qquad \vdots \\ \boldsymbol{a}_m &= \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix}, \end{aligned}$$

and think of A as $\begin{bmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_m \end{bmatrix}$. We call $\boldsymbol{a}_i$ the i-th row vector of A.

Formula 2. Let $A = \begin{bmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_m \end{bmatrix}$ be as above. Let B be any matrix whose number of rows is r. Then $AB = \begin{bmatrix} \boldsymbol{a}_1 B \\ \boldsymbol{a}_2 B \\ \vdots \\ \boldsymbol{a}_m B \end{bmatrix}$.

Example 9. Look at the same matrix multiplication as in Example 8.

The first row of the resulting matrix is $\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$.

The second row of the resulting matrix is $\begin{bmatrix} a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$.

- **Systems of linear equations in matrix form.**

Consider the system of linear equations

$$\begin{cases} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1, \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2, \\ \qquad \cdots \qquad \cdots \qquad \cdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = b_m \end{cases}$$

where a_{ij} and b_i are scalars $\big($constants$\big)$. The same system can be rewritten as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

In other words, let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

and the above is

$$A\boldsymbol{x} = \boldsymbol{b}.$$

- **Homogeneous system of linear equations.**

  Consider the system

$$\begin{cases} a_{11}\,x_1 + a_{12}\,x_2 + \cdots + a_{1n}\,x_n = 0, \\ a_{21}\,x_1 + a_{22}\,x_2 + \cdots + a_{2n}\,x_n = 0, \\ \qquad \cdots \qquad \cdots \qquad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0. \end{cases}$$

Note the right-hand side on each line equals 0. This is called a homogeneous system.
The same system can be rewritten as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

In other words, letting A and $\boldsymbol{x}$ be as above, and $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, the same system can be rewritten as

$$A\boldsymbol{x} = \mathbf{0}.$$

Example 10. The system of linear equations

$$\begin{cases} \quad\quad\quad 6x_2 + 4x_3 + 2x_4 = -1, \\ 3x_1 + 3x_2 \quad\quad\quad + 7x_4 = 4, \\ 2x_1 \quad\quad - 3x_3 \quad\quad\quad = 10 \end{cases}$$

is rewritten as

$$\begin{bmatrix} 0 & 6 & 4 & 2 \\ 3 & 3 & 0 & 7 \\ 2 & 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 10 \end{bmatrix}.$$

Exercise 7. Rewrite the system of linear equations

$$\begin{cases} x_1 - x_2 - x_3 - x_4 = -2, \\ -x_1 + x_2 - x_3 - x_4 = -2, \\ -x_1 - x_2 + x_3 - x_4 = -2, \\ -x_1 - x_2 - x_3 + x_4 = -2. \end{cases}$$

in the form $A\boldsymbol{x} = \boldsymbol{b}$.

Exercise 8. Rewrite the system of linear equations

$$\begin{cases} 4x_1 \quad\quad - 3x_3 \quad\quad + 5x_5 = 0, \\ 2x_1 - 2x_2 \quad\quad + x_4 \quad\quad = 0, \\ x_1 + 6x_2 + 8x_3 \quad\quad + x_5 = 0, \\ 3x_1 - x_2 + 4x_3 - x_4 - 7x_5 = 0, \\ x_1 + x_2 + x_3 + x_4 + x_5 = 0 \end{cases}$$

in the form $A\boldsymbol{x} = \mathbf{0}$.