Math 290 ELEMENTARY LINEAR ALGEBRA

REVIEW OF LECTURES – XVI

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Let's start with an example.

Example 5.

Let's consider

$$A = \begin{bmatrix} -2 & -14 \\ 3 & 11 \end{bmatrix}.$$

The following may be out of the blue, but bear with me. This matrix A is clearly *not* a diagonal matrix. However, form PAQ as follows:

$$(*) \qquad \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}}_{\substack{\| \\ P}} \underbrace{\begin{bmatrix} -2 & -14 \\ 3 & 11 \end{bmatrix}}_{\substack{\| \\ A}} \underbrace{\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}}_{\substack{\| \\ Q}}.$$

Here the choices of P and Q are deliberate. Don't ask me what prompted me to make these choices for P and Q. All I can say right now is, with these choices of P and Q, once we calculate this PAQ, it will become

$$PAQ = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}.$$

Is there anything that stands out? Yes, this last matrix is a <u>diagonal matrix</u>.

But that's not the end of it. P and Q are actually <u>inverses of each other</u>. Indeed, let's form PQ:

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}}_{\substack{\| \\ P}} \underbrace{\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}}_{\substack{\| \\ Q}}.$$

Once you calculate this, you will end up getting

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,$$

the identity matrix. So in other words, $Q = P^{-1}$. So, PAQ is PAP^{-1}. Now, this way we arrive at the following:

$$\boxed{\begin{array}{l} A = \begin{bmatrix} -2 & -14 \\ 3 & 11 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \\ \Longrightarrow \qquad PAP^{-1} = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}. \end{array}}$$

So again, A is not diagonal but PAP^{-1} is.

In the above, I didn't explain how I came up with P and Q, where one is the inverse of the other. Let me oblige. First, breaking news:

Fact.
$$A = \begin{bmatrix} -2 & -14 \\ 3 & 11 \end{bmatrix}$$

(the same A as above) has 4 and 5 as its eigenvalue. This is due to the fact that 4 and 5 are the two diagonal entries of the 'diagonalized' matrix PAP^{-1}.

An obvious question here is "how come?" Can anyone explain? Believe me or not, the theory that we have been thoroughly developing is tailor-made to this situation. We have been patiently building up some foundations, and this is the place to readily apply them. Okay, here is the clue:

Clue. Agree

$$P\Big(\lambda I - A\Big)P^{-1} \;=\; \lambda I \;-\; PAP^{-1}$$

$$= \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} \lambda-4 & 0 \\ 0 & \lambda-5 \end{bmatrix}.$$

In short, $P\Big(\lambda I - A\Big)P^{-1} = \begin{bmatrix} \lambda-4 & 0 \\ 0 & \lambda-5 \end{bmatrix}$. Let's take the determinant:

(@) $$\det\left(P\Big(\lambda I - A\Big)P^{-1}\right) = \begin{vmatrix} \lambda-4 & 0 \\ 0 & \lambda-5 \end{vmatrix}.$$

1. First, the right-hand side of (@) is clearly $\Big(\lambda-4\Big)\Big(\lambda-5\Big)$.

2. Second, by Product Formula ("Review of Lectures – IV"), the left-hand side of (@) is broken up as

$$\Big(\det P\Big)\left(\det\Big(\lambda I - A\Big)\right)\Big(\det P^{-1}\Big) = \Big(\det P\Big)\left(\det\Big(\lambda I - A\Big)\right)\frac{1}{\det P}$$

$$= \Big(\det P\Big)\frac{1}{\det P}\left(\det\Big(\lambda I - A\Big)\right)$$

[here $\det\Big(\lambda I - A\Big)$ and $\dfrac{1}{\det P}$ are interchangeable because both are scalars]

$$= \det\Big(\lambda I - A\Big)$$

$$= \chi_A(\lambda) \quad \Big(\text{the characteristic polynomial of } A\Big).$$

From 1. and 2. above, we conclude $\chi_A(\lambda) = \Big(\lambda-4\Big)\Big(\lambda-5\Big)$. $\square$

- Let me recite 'Fact' below:

Fact.

$$A = \begin{bmatrix} -2 & -14 \\ 3 & 11 \end{bmatrix}$$

(the same A as above) has 4 and 5 as its eigenvalue. This is due to the fact 4 and 5 are the two diagonal entries of the 'diagonalized' matrix PAP^{-1}:

$$PAP^{-1} = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}.$$

Next, here is another fact:

Fact 2. In

$$Q = P^{-1} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix},$$

the two columns are the so-called "eigenvectors", namely, the vector $\boldsymbol{x}$ satisfying $A\boldsymbol{x} = \lambda\boldsymbol{x}$ where λ is one of the eigenvalues of A.

- $\begin{bmatrix} 7 \\ -3 \end{bmatrix}$ is an eigenvector of A associated with the eigenvalue $\lambda = 4$.
- $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector of A associated with the eigenvalue $\lambda = 5$.

- Let's verify these:

$$\begin{bmatrix} -2 & -14 \\ 3 & 11 \end{bmatrix}\begin{bmatrix} 7 \\ -3 \end{bmatrix} = \begin{bmatrix} (-2)\cdot 7 + (-14)\cdot(-3) \\ 3\cdot 7 + 11\cdot(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 28 \\ -12 \end{bmatrix} = 4\begin{bmatrix} 7 \\ -3 \end{bmatrix},$$

$$\begin{bmatrix} -2 & -14 \\ 3 & 11 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} (-2)\cdot(-2) + (-14)\cdot 1 \\ 3\cdot(-2) + 11\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} -10 \\ 5 \end{bmatrix} = 5\begin{bmatrix} -2 \\ 1 \end{bmatrix}. \qquad \square$$

• But the real question is why the matrix Q, whose columns are eigenvectors of A, has the ability to make $Q^{-1}AQ \ \left(= \ PAP^{-1} \right)$ diagonal.

The answer is very simple. It is as follows. For simplicity, let's denote

$$\boldsymbol{x} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}, \qquad \boldsymbol{y} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

So

- ○ $\boldsymbol{x}$ is an eigenvector of A associated with the eigenvalue $\lambda = 4$,
- ○ $\boldsymbol{y}$ is an eigenvector of A associated with the eigenvalue $\lambda = 5$,

and moreover $Q = \begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \end{bmatrix}$. Then

$$\begin{aligned} AQ &= A \begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \end{bmatrix} \\ &= \begin{bmatrix} A\boldsymbol{x} & A\boldsymbol{y} \end{bmatrix} \\ &= \begin{bmatrix} 4\,\boldsymbol{x} & 5\,\boldsymbol{y} \end{bmatrix}. \end{aligned}$$

Here, the matrix $\begin{bmatrix} 4\,\boldsymbol{x} & 5\,\boldsymbol{y} \end{bmatrix}$ is actually rewritten as

$$\begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}.$$

$\Bigg($Indeed, physically calculate $\begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}$ and $\begin{bmatrix} 4p & 5q \\ 4r & 5s \end{bmatrix}$ comes out.$\Bigg)$

Here, remember $\begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \end{bmatrix} = Q$. So, in short,

$$AQ = Q \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}.$$

Multiply Q^{-1} from the left to the both sides of this last identity, and we obtain

$$Q^{-1}AQ = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}. \qquad \square$$

• We can now extrapolate the above picture, and establish a method for diagonalizing a given matrix A.

Recipé to diagonalize a given matrix.

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Suppose A has <u>two distinct</u> eigenvalues

$$\chi_A(\lambda) = \left(\lambda - \lambda_1\right)\left(\lambda - \lambda_2\right) \qquad \left(\lambda_1 \neq \lambda_2\right).$$

Suppose

○ $\boldsymbol{x} = \begin{bmatrix} p \\ r \end{bmatrix}$ is an eigenvector of A associated with the eigenvalue $\lambda = \lambda_1$.

○ $\boldsymbol{y} = \begin{bmatrix} q \\ s \end{bmatrix}$ is an eigenvector of A associated with the eigenvalue $\lambda = \lambda_2$.

Then set

$$Q = \begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}.$$

Then

$$Q^{-1}AQ = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

In other words, set $P = Q^{-1}$, and

$$PAP^{-1} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Example 6. Let's diagonalize the matrix

$$A = \begin{bmatrix} -4 & 6 \\ 7 & -5 \end{bmatrix}.$$

Step 1. Find the eigenvalues. This is routine:

$$\begin{aligned} \chi_A(\lambda) &= \det\left(\lambda I - A\right) \\ &= \begin{vmatrix} \lambda+4 & -6 \\ -7 & \lambda+5 \end{vmatrix} \\ &= \left(\lambda+4\right)\left(\lambda+5\right) - \left(-6\right)\cdot\left(-7\right) \\ &= \lambda^2 + 9\lambda + 20 - 42 \\ &= \lambda^2 + 9\lambda - 22 \\ &= \left(\lambda-2\right)\left(\lambda+11\right). \end{aligned}$$

So, the eigenvalues of A are

$$\lambda = 2 \qquad \text{and} \qquad \lambda = -11.$$

Step 2. Find eigenvectors of A associated with each of the two eigenvalues of A (**Step 2a** and **Step 2b** below).

Step 2a. Find an eigenvector of A associated with $\lambda = 2$.

Since $A = \begin{bmatrix} -4 & 6 \\ 7 & -5 \end{bmatrix}$, the equation $A\boldsymbol{x} = 2\boldsymbol{x}$ is

(@) $$\begin{bmatrix} -4 & 6 \\ 7 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}.$$

That is,

$$\begin{bmatrix} -4x + 6y \\ 7x - 5y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}.$$

That is,

$$\begin{cases} -4x + 6y = 2x, \\ 7x - 5y = 2y. \end{cases}$$

Shift the terms:

$$\begin{cases} -6x + 6y = 0, \\ 7x - 7y = 0. \end{cases}$$

So two essentially identical equations came out. These equations are the same as

$$\boxed{x - y = 0.}$$

Clearly

$$x = 1, \;\; y = 1$$

works. Thus:

- $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A associated with the eigenvalue $\lambda = 2$.

Step 2b. Find an eigenvector of A associated with $\lambda = -11$.

Since $A = \begin{bmatrix} -4 & 6 \\ 7 & -5 \end{bmatrix}$, the equation $A\boldsymbol{x} = -11\boldsymbol{x}$ is

(@)
$$\begin{bmatrix} -4 & 6 \\ 7 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -11 \begin{bmatrix} x \\ y \end{bmatrix}.$$

That is,

$$\begin{bmatrix} -4x + 6y \\ 7x - 5y \end{bmatrix} = \begin{bmatrix} -11x \\ -11y \end{bmatrix}.$$

That is,

$$\begin{cases} -4x + 6y = -11x, \\ 7x - 5y = -11y. \end{cases}$$

Shift the terms:

$$\begin{cases} 7x + 6y = 0, \\ 7x + 6y = 0. \end{cases}$$

So two identical equations came out. Delete one of them:

$$\boxed{7x + 6y = 0.}$$

Clearly

$$x = 6, \ \ y = -7$$

works. Thus:

- $\boldsymbol{y} = \begin{bmatrix} 6 \\ -7 \end{bmatrix}$ is an eigenvector of A associated with the eigenvalue $\lambda = -11$.

Step 3. Form

$$Q = \begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \end{bmatrix},$$

using $\boldsymbol{x}$ from Step 2a, and $\boldsymbol{y}$ from Step 2b:

$$Q = \begin{bmatrix} 1 & 6 \\ 1 & -7 \end{bmatrix}.$$

Answer. $A = \begin{bmatrix} -4 & 6 \\ 7 & -5 \end{bmatrix}$ is diagonalized as follows:

$$Q^{-1}AQ = \begin{bmatrix} 2 & 0 \\ 0 & -11 \end{bmatrix}, \qquad \text{where} \qquad Q = \begin{bmatrix} 1 & 6 \\ 1 & -7 \end{bmatrix}.$$

- Just in case, let's find $Q^{-1} = P$. First, note

$$\det Q = \begin{vmatrix} 1 & 6 \\ 1 & -7 \end{vmatrix}$$

$$= 1 \cdot \big(-7\big) - 6 \cdot 1 = -13.$$

So

$$P = \begin{bmatrix} 1 & 6 \\ 1 & -7 \end{bmatrix}^{-1} = \frac{1}{\det Q} \begin{bmatrix} -7 & -6 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{-13} \begin{bmatrix} -7 & -6 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{13} \begin{bmatrix} 7 & 6 \\ 1 & -1 \end{bmatrix}.$$

Accordingly, we may paraphrase our answer as follows:

Alternative Answer. $A = \begin{bmatrix} -4 & 6 \\ 7 & -5 \end{bmatrix}$ is diagonalized as follows:

$$PAP^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & -11 \end{bmatrix}, \qquad \text{where} \qquad P = \frac{1}{13} \begin{bmatrix} 7 & 6 \\ 1 & -1 \end{bmatrix}.$$

Note. You may instead throw $P = \begin{bmatrix} 7 & 6 \\ 1 & -1 \end{bmatrix}$ in "Alternative Answer". This is acceptable, as long as you are aware of the fact that this P and Q above are no longer inverses of each other. Indeed, when you replace P with a non-zero scalar multiple of P, it does not affect PAP^{-1}.