Math 290 ELEMENTARY LINEAR ALGEBRA

REVIEW OF LECTURES – II

August 23 (Wed), 2017

Instructor: Yasuyuki Kachi

Line #: 25751.

§2. Determinants – Intro.

A quick run down:

(1) Linear algebra at the outset studies systems of linear equations ('system' in what follows).

(2) Along the way, matrices enter the scene. (More on this today.)

(3) Solution formula exists, provided a certain condition is met: "$\Delta \neq 0$", where Δ is called a 'determinant' (see (4) below).

(4) The shape of the formula prompts us to isolate the notion of determinants. The formula can be neatly spelt out using determinants. The denominators of the expression of x, y (or x_1, x_2, $\cdots$) in the formula are all one single determinant Δ.

(5) The determinant formation exhibits some glaring patterns. Discerning and dissecting such patterns is at the epitome of linear algebra (elaboration pending, technical term here is 'multi-linearity'). That is so, (6) below notwithstanding.

(6) A nuisance: Already for the 3×3 case, the determinant expression is long(ish). Much more so for their 4×4 counterparts, and you can imagine how that goes for 5×5 and larger.

(7) It is natural to ask whether one can do away with the determinants (bypass the formula) when it comes to solving a system.

(8) Yes, that is indeed feasible — there is such a thing called Gaussian elimination method (I only threw the name).

(9) However, the gist of that method essentially amounts to evaluating determinants, so it's Catch 22.

(10) In sum, determinants are the nucleus of linear algebra. The progress of the lecture will be centered around them.

(11; Side-track) I have briefly touched on the issue whether computers can handle all math problems $\big($thus, by implication, the issue whether math research is obsolete$\big)$. My answer: 'Hardly'. My take: There exist a myriad of outstanding open problems in math. The cutting-edge math research goes around them. Most importantly, we *human* mathematicians are in charge. The geniuses $\big($top-dogs$\big)$ among us shoot for claiming the last word $\big($= the ultimate solution to the question to which everything else boils down$\big)$. Others pave the way towards the last word while staying within the firing distance, lurking to strike. We also constantly keep adding new $\big($meaningful and important$\big)$ open problems and new insights to that bucket-list. This is how math thrives and evolves. This is how math secures a spot as an outright exuberant corner of science. $\big($Will find another time to elaborate more.$\big)$

Are we on the same page? All right. Let's jump-start today's lesson.

- The first order of business: Let's officially define the determinant:

Definition (Determinant; 2 × 2).

The determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is defined as follows:

$$\boxed{\quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \quad}$$

— So, for example:

Example 1.

$$\begin{vmatrix} 7 & 5 \\ 2 & 1 \end{vmatrix} = 7 \cdot 1 - 5 \cdot 2$$

$$= -3.$$

Not that hard. However, there is something I want to stress:

$$\underline{\underline{\textit{Determinant is defined for each matrix,}}}$$

meaning:

$$\boxed{\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad \underline{\underline{\text{is the determinant of the matrix}}} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}.}$$

For example, $\begin{vmatrix} 7 & 5 \\ 2 & 1 \end{vmatrix} = -3$ $\big($as we have just calculated$\big)$ is regarded as the determinant of the matrix $\begin{bmatrix} 7 & 5 \\ 2 & 1 \end{bmatrix}$. By implication: For your successful grasp of the concept of determinants, you need to agree on the following first and foremost:

- *First there is this notion of matrices.*
- *Then the determinant is defined for each* $\big(2 \times 2\big)$ *matrix.*
- *Matrices themselves are arrays, whereas*:
- *The determinant of a matrix is a scalar.*

Here, 'scalar' means a single number. -3 is a scalar. $\begin{bmatrix} 7 & 5 \\ 2 & 1 \end{bmatrix}$ is not a scalar.

• It is common practice in linear algebra to use a letter, typically a capital letter, to represent a matrix. So you can say

$$\text{“Let } A \text{ stand for the matrix } \begin{bmatrix} a & b \\ c & d \end{bmatrix}.\text{”}$$

Or you can just say

$$\text{“Let} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.\text{”}$$

Let's streamline everything, taking all the above into account, and let's make the following *official* definition of the determinant:

Official Definition of Determinant $\mathbf{(2 \times 2)}$.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Define <u>the determinant of the matrix A</u> as

$$\boxed{\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.}$$

Example 2. (1) For $A = \begin{bmatrix} -6 & 2 \\ 8 & -4 \end{bmatrix}$, its determinant is

$$\det A = \begin{vmatrix} -6 & 2 \\ 8 & -4 \end{vmatrix} = (-6)\cdot(-4) - 2\cdot 8$$

$$= 8.$$

(2) For $A = \begin{bmatrix} -2 & 4 \\ -3 & 6 \end{bmatrix}$, its determinant is

$$\det A = \begin{vmatrix} -2 & 4 \\ -3 & 6 \end{vmatrix} = (-2)\cdot 6 - 4\cdot(-3)$$

$$= 0.$$

(3) For $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, its determinant is

$$\det A = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1\cdot 1 - 0\cdot 0$$

$$= 1.$$

Exercise 1. Calculate:

(1) $\begin{vmatrix} 1 & 6 \\ 1 & 3 \end{vmatrix}$. (2) $\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}$. (3) $\begin{vmatrix} 2 & 5 \\ \frac{3}{10} & 4 \end{vmatrix}$.

(4) $\det A$, where $A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}$.

(5a) $\det A$, where $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. (5b) $\det B$, where $B = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

(6a) $\det A$, where $A = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$. (6b) $\det B$, where $B = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$.

(7a) $\quad \det A, \quad$ where $\quad A = \begin{bmatrix} -1+\sqrt{5} & -\sqrt{10+2\sqrt{5}} \\ \sqrt{10+2\sqrt{5}} & -1+\sqrt{5} \end{bmatrix}.$

(7b) $\quad \det B, \quad$ where $\quad B = \begin{bmatrix} \frac{-1+\sqrt{5}}{4} & \frac{-\sqrt{10+2\sqrt{5}}}{4} \\ \frac{\sqrt{10+2\sqrt{5}}}{4} & \frac{-1+\sqrt{5}}{4} \end{bmatrix}.$

- Now, keeping the narrative intact, let's define the 3×3 determinant:

Official Definition of Determinant (3×3).

Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$. Define <u>the determinant of the matrix A</u> as

$$\det A = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1.$$

- So, while the 2×2 one wasn't too bad, the 3×3 one is longish, like I said earlier. Now, though I don't want to get ahead of myself, let me just throw one thing. If you stare at the above 3×3 determinant, don't you realize

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1$$

$$= a_1 \Big(b_2 c_3 - b_3 c_2\Big) - a_2 \Big(b_1 c_3 - b_3 c_1\Big) + a_3 \Big(b_1 c_2 - b_2 c_1\Big)$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

But here is what's more. Each of the following six lines is called the co-factoring of the 3×3 determinant:

(i)$_a$
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

(i)$_b$
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}.$$

(i)$_c$
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

(ii)$_1$
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}.$$

(ii)$_2$
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}.$$

(ii)$_3$
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

Exercise 2. Compare these six lines and discern their patterns $\big($including the signs that come with the terms$\big)$.

So what do all these entail? Yes, even though at a first glance the expression of the 3×3 determinant is long(ish), it actually transpires to be formed through the 2×2 determinants. And this is indeed a part of the bigger picture. The above is a snapshot of some <u>hierarchical structure</u> existing among the expressions of different size determinants $\big(2 \times 2;\ 3 \times 3;\ 4 \times 4;\ \cdots\big)$. So, even though your first impression might have been that the determinant business is *ad nausium*, it actually has some meat to it. Full analysis of all this — hierarchial structure of the determinants — is our first goal this semester. So this isn't too bad altogether. All right?

Example 3. For $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -2 \\ 3 & -1 & 4 \end{bmatrix}$, let's calculate $\det A = \begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & -2 \\ 3 & -1 & 4 \end{vmatrix}$.

We may directly apply the definition of the determinant:

$$\begin{aligned} \det A \ &= \begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & -2 \\ 3 & -1 & 4 \end{vmatrix} \\ &= 1 \cdot 1 \cdot 4 - 1 \cdot \big(-2\big) \cdot \big(-1\big) - 2 \cdot 0 \cdot 4 + 2 \cdot \big(-2\big) \cdot 3 + 2 \cdot 0 \cdot \big(-1\big) - 2 \cdot 1 \cdot 3 \\ &= 4 - 2 - 0 + \big(-12\big) + 0 - 6 \ = \ -16. \end{aligned}$$

But we could've applied the co-factoring, say $(\mathrm{i})_a$, for the same problem instead:

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & -2 \\ 3 & -1 & 4 \end{vmatrix} \ &= \ 1 \cdot \begin{vmatrix} 1 & -2 \\ -1 & 4 \end{vmatrix} - 2 \cdot \begin{vmatrix} 0 & -2 \\ 3 & 4 \end{vmatrix} + 2 \cdot \begin{vmatrix} 0 & 1 \\ 3 & -1 \end{vmatrix} \\ &= \ 1 \cdot 2 - 2 \cdot 6 + 2 \cdot \big(-3\big) \ = \ -16. \end{aligned}$$

Or, we could've applied a different co-factoring, say $(\mathrm{ii})_2$ instead:

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & -2 \\ 3 & -1 & 4 \end{vmatrix} \ &= \ -2 \cdot \begin{vmatrix} 0 & -2 \\ 3 & 4 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} - \big(-1\big) \cdot \begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} \\ &= \ -2 \cdot 6 + 1 \cdot \big(-2\big) - \big(-1\big) \cdot \big(-2\big) \ = \ -16. \end{aligned}$$

Exercise 3. Calculate:

(1) $\begin{vmatrix} 5 & 6 & 1 \\ 1 & 3 & -4 \\ 2 & 5 & 2 \end{vmatrix}$. (2) $\begin{vmatrix} 1 & -1 & 0 \\ 3 & 1 & 1 \\ -2 & 2 & 2 \end{vmatrix}$. (3) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$.

(4) $\det A$, where $A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$.

(5) $\det A$, where $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$.

Factor the answer for (5).

(6) $\det A$, where $A = \begin{bmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{bmatrix}$.

(7) $\det A$, where $A = \begin{bmatrix} 1 & x & x^2 \\ x & 1 & x^3 \\ x^2 & x^3 & 1 \end{bmatrix}$.

Factor the answer for (7).

(8)* $\det A$, where

$$A = \begin{bmatrix} a^2-b^2-c^2+d^2 & 2\Big(ab+cd\Big) & 2\Big(-ac+bd\Big) \\ 2\Big(-ab+cd\Big) & a^2-b^2+c^2-d^2 & 2\Big(ad+bc\Big) \\ 2\Big(ac+bd\Big) & 2\Big(-ad+bc\Big) & a^2+b^2-c^2-d^2 \end{bmatrix}.$$

Factor the answer for (8).