Math 290 ELEMENTARY LINEAR ALGEBRA

REVIEW OF LECTURES – III

August 28 (Mon), 2017

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§3. Matrix arithmetic — I. Inverse of a matrix.

• Today's agenda: Matrix arithmetic. Ready? Do you guys remember the following from Day 1?

$$(*) \quad \boxed{\left\{ \begin{aligned} 4x + 3y &= 5, \\ 2x - 6y &= -7 \end{aligned} \right.} \quad \underset{\text{“equivalent”}}{\Longleftrightarrow} \quad \boxed{\begin{bmatrix} 4 & 3 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -7 \end{bmatrix}}$$

This means that the box on the right is a mere paraphrase of the box on the left. I want you to focus on the one in the <u>right</u>. Let's give names for the objects in sight:

$$\underbrace{\begin{bmatrix} 4 & 3 \\ 2 & -6 \end{bmatrix}}_{\substack{\| \\ A}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\substack{\| \\ \boldsymbol{x}}} = \underbrace{\begin{bmatrix} 5 \\ -7 \end{bmatrix}}_{\substack{\| \\ \boldsymbol{b}}}$$

So the equation is basically

$$\boxed{A\boldsymbol{x} = \boldsymbol{b}.}$$

Not for nothing, let's <u>pretend</u> that, in this last equation, all the letters are numbers. How would you solve it for $\boldsymbol{x}$ $\big($the unknown$\big)$? Yes. Just like

$$3x = 1 \quad \underset{\text{solve}}{\Longrightarrow} \quad x = \frac{1}{3} = 3^{-1},$$

$$\pi x = \sqrt{2} \quad \underset{\text{solve}}{\Longrightarrow} \quad x = \frac{\sqrt{2}}{\pi} = \pi^{-1} \cdot \sqrt{2},$$

we would solve $A\boldsymbol{x} = \boldsymbol{b}$ as

$$\text{“}\ \boldsymbol{x} = \frac{\boldsymbol{b}}{A}\ \text{”}$$

or

$$\text{“}\ \boldsymbol{x} = A^{-1}\boldsymbol{b}\ \text{”},$$

with quote-unquote “ ”. However, in reality, A is not a number. A is a matrix. So the division $\dfrac{\boldsymbol{b}}{A}$ does not make sense as it stands. But we might still be able to make sense of A^{-1}, as a matrix, in such a way that $\boldsymbol{x} = A^{-1}\boldsymbol{b}$ is indeed the correct answer for the equation.

- Let's cut to the chase again: I hereby share the following information:

  The fraction $\dfrac{\boldsymbol{b}}{A}$ *does not quite make sense. However, good news*: A^{-1} *makes sense, as a* 2×2 *matrix, and thus* $A^{-1}\boldsymbol{b}$ *also makes sense, under one condition*: $\det A \neq 0$.

So,

$$\boxed{A\boldsymbol{x} = \boldsymbol{b} \quad \underset{\substack{\text{can solve,} \\ \text{if } \det A \neq 0}}{\Longrightarrow} \quad \boldsymbol{x} = A^{-1}\boldsymbol{b}.}$$

This is a legit way to solve the equation

$$\boxed{A\boldsymbol{x} = \boldsymbol{b}.}$$

Most importantly, I must tell you how to form A^{-1} out of A:

Inverse of a 2×2 matrix.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The <u>inverse</u> A^{-1} of A is the following matrix:

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}.$$

A^{-1} exists, provided $\det A = ad - bc \neq 0$.

— All right, let's dissect this:

$$\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}.$$

A little fuss — this matrix is so crammed, because it is made of fractions. Those fractions aren't random, though. On second look, realize that the denominators are all $ad - bc$, which is nothing else but $\det A$. So, we might as well write this as something like

$$\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \qquad \text{or} \qquad \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Technically, though, we need to "validate" that. Here is what I mean: Agree that

- the part $\dfrac{1}{ad - bc}$ $\left(= \dfrac{1}{\det A} \right)$ is a scalar,

whereas

- the part $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ is a matrix.

We have juxtaposed these two ingredients, and we mean it to signify

$$\text{“ } \underline{\underline{\text{a scalar being multiplied to}}} \text{ a matrix ”.}$$

The problem is, we haven't officially implemented such an operation in this class yet. Technically speaking, no matter how natural it is that

$$10 \begin{bmatrix} 8 & 3 \\ 2 & 7 \end{bmatrix} \qquad \text{means} \qquad \begin{bmatrix} 80 & 30 \\ 20 & 70 \end{bmatrix},$$

$$\frac{1}{5} \begin{bmatrix} 2 & -4 \\ -1 & 1 \end{bmatrix} \qquad \text{means} \qquad \begin{bmatrix} \frac{2}{5} & \frac{-4}{5} \\ \frac{-1}{5} & \frac{1}{5} \end{bmatrix},$$

etc., we still need to *officially* declare that

$$s \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{means} \qquad \begin{bmatrix} sa & sb \\ sc & sd \end{bmatrix}.$$

Nothing really stops us from making such a declaration. Setting such a rule is universally adopted. So, here we go, an official declaration of the rule:

- **Definition (Scalar multiplied to a matrix).** Let s be a scalar. Then

$$\boxed{s \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} sa & sb \\ sc & sd \end{bmatrix}.}$$

Paraphrase:

$$\boxed{\begin{array}{l} \text{If} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{and} \qquad s : \text{ a scalar} \\ \qquad \Longrightarrow \qquad sA = \begin{bmatrix} sa & sb \\ sc & sd \end{bmatrix}. \end{array}}$$

Example 1. (1)
$$3\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} = \begin{bmatrix}3 & 6\\ 9 & 12\end{bmatrix}.$$

(2)
$$4\begin{bmatrix}3 & 3\\ 3 & 3\end{bmatrix} = \begin{bmatrix}12 & 12\\ 12 & 12\end{bmatrix}.$$

(3)
$$\frac{1}{7}\begin{bmatrix}5 & 7\\ -1 & 0\end{bmatrix} = \begin{bmatrix}\frac{5}{7} & 1\\ \frac{-1}{7} & 0\end{bmatrix}.$$

(4)
$$\frac{9}{2}\begin{bmatrix}\frac{2}{9} & 2\\ \frac{4}{9} & \frac{1}{9}\end{bmatrix} = \begin{bmatrix}1 & 9\\ 2 & \frac{1}{2}\end{bmatrix}.$$

(5)
$$1\begin{bmatrix}0 & -2\\ 6 & 3\end{bmatrix} = \begin{bmatrix}0 & -2\\ 6 & 3\end{bmatrix}.$$

- An obvious generalization of (5) is

$$\boxed{1\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}.}$$

Paraphrase:

$$\boxed{\text{If} \quad A = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \quad \Longrightarrow \quad 1A = A.}$$

- **Definition (negation).**

$$\boxed{-\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}-a & -b\\ -c & -d\end{bmatrix}.}$$

Paraphrase:

$$\boxed{\text{If} \quad A = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \quad \Longrightarrow \quad -A = \begin{bmatrix}-a & -b\\ -c & -d\end{bmatrix}.}$$

Example 2. $0\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}, \qquad 8\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}.$

- An obvious generalization of Example 2 is

$$0\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}, \qquad s\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}.$$

- We denote $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ as O. Then we can paraphrase it as:

$$\boxed{\begin{array}{l} \text{If} \quad A = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \quad \text{and} \quad s : \text{a scalar} \\ \Longrightarrow \quad 0A = O, \quad sO = O. \end{array}}$$

Example 3a. $(-1)\begin{bmatrix}3 & 4\\ 5 & 9\end{bmatrix} = \begin{bmatrix}-3 & -4\\ -5 & -9\end{bmatrix}.$

Example 3b. $-\begin{bmatrix}3 & 4\\ 5 & 9\end{bmatrix} = \begin{bmatrix}-3 & -4\\ -5 & -9\end{bmatrix}.$

- As you can clearly see, the negative of a matrix and the (-1) times the same matrix are always equal:

$$\boxed{(-1)\begin{bmatrix}a & b\\ c & d\end{bmatrix} = -\begin{bmatrix}a & b\\ c & d\end{bmatrix}.}$$

Paraphrase:

$$\boxed{\text{If} \quad A = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \quad \Longrightarrow \quad (-1)A = -A.}$$

Exercise 1. Write each of the following in the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

(1) $3\begin{bmatrix} -4 & 2 \\ 6 & 5 \end{bmatrix}$. (2) $\frac{1}{2}\begin{bmatrix} 10 & 12 \\ 8 & 4 \end{bmatrix}$. (3) $\frac{1}{8}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(4) $\left(-2\right)\begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}$. (5) $1\begin{bmatrix} 7 & -5 \\ \frac{1}{2} & 1 \end{bmatrix}$. (6) $0\begin{bmatrix} 124 & 242 \\ 163 & 89 \end{bmatrix}$.

(7) $1000\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Exercise 2. Write each of the following in the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

(1) $-\begin{bmatrix} -6 & -8 \\ 3 & 4 \end{bmatrix}$. (2) $-\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. (3) $-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Exercise 3. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, define

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \qquad \left(\text{the } \underline{\text{ transpose }} \text{ of } A\right).$$

Assume $A^T = -A$. Prove that there is a scalar s such that

$$A = s\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

• We were originally talking about the inverse of a matrix. We then got side-tracked a bit along the way. So, back to page 3:

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} .$$

Now that we have established the concept "a scalar multipled to a matrix", we are officially entitled to paraphrase the definition of the inverse in page 3:

Inverse of a 2×2 matrix, paraphrased.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The inverse A^{-1} of A is the following matrix:

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

A^{-1} exists, provided $\det A = ad - bc \neq 0$.

• **Adjoint matrix.**

For convenience of reference, we give it a name for a part of the A^{-1} formation:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies A^{-1} = \frac{1}{\det A} \underbrace{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}_{\substack{\| \\ \text{adj } A}}$$

So,

$$\boxed{A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \Longrightarrow \qquad \text{adj}\, A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.}$$

We call adj A the <u>adjoint matrix</u> of A.

- We may accordingly further paraphrase the above:

Inverse of a 2×2 matrix, paraphrased — II.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The <u>inverse</u> A^{-1} of A is the following matrix:

$$\boxed{A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det A}\, \text{adj}\, A,}$$

where

$$\det A = ad - bc, \qquad \text{and}$$

$$\text{adj}\, A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

A^{-1} exists, provided $\det A = ad - bc \neq 0$.

- Let's calculate A^{-1} for some concete matrix A.

Example 4. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$. Let's find A^{-1}.

Here is how it goes:

Step 1. Calculate the determinant of A:

$$\det A = \begin{vmatrix} 2 & 3 \\ 4 & 7 \end{vmatrix} = 2 \cdot 7 - 3 \cdot 4$$

$$= 2.$$

Step 2. Form the adjoint of A:

$$\text{adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -3 \\ -4 & 2 \end{bmatrix}.$$

Step 3. Finish it off:

$$A^{-1} = \frac{1}{\det A} \text{ adj } A$$

$$= \frac{1}{2} \begin{bmatrix} 7 & -3 \\ -4 & 2 \end{bmatrix}.$$

You may write the answer as

$$\begin{bmatrix} \frac{7}{2} & \frac{-3}{2} \\ -2 & 1 \end{bmatrix},$$

which is optional.

Example 5. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -5 & 3 \end{bmatrix}$. Let's find A^{-1}.

Here is how it goes:

Step 1. Calculate the determinant of A:

$$\det A = \begin{vmatrix} 3 & -2 \\ -5 & 3 \end{vmatrix} = 3 \cdot 3 - \left(-2\right) \cdot \left(-5\right)$$

$$= -1.$$

Step 2. Form the adjoint of A:

$$\text{adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}.$$

Step 3. Finish it off:

$$A^{-1} = \frac{1}{\det A} \text{ adj } A$$

$$= \frac{1}{-1} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix}.$$

- **What if the determinant of A equals 0 ?**

A natural question arises. What if $\det A = 0$ for a given matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$? What can one say about A^{-1}?

— The answer is simple: In such a case, the inverse A^{-1} does not exist.

Example 6. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 1 & 2 \end{bmatrix}$. Let's decide whether A^{-1} exists.

For that matter, it suffices to calculate the determinant of A:

$$\det A = \begin{vmatrix} 4 & 8 \\ 1 & 2 \end{vmatrix} = 4 \cdot 2 - 8 \cdot 1$$

$$= 0.$$

So, we conclude that A^{-1} does not exist.

Exercise 4. Decide whether the inverse A^{-1} of A exists, in each of (1–12) below. If it does, then calculate it.

(1) $A = \begin{bmatrix} 5 & 7 \\ -1 & 3 \end{bmatrix}$.
(2) $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.
(3) $A = \begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix}$.

(4) $A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$.
(5) $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.
(6) $A = \begin{bmatrix} 1 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{9} \end{bmatrix}$.

(7) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
(8) $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.
(9) $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(10) $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.
(11) $A = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$.

(12) $A = \begin{bmatrix} \frac{-1+\sqrt{5}}{4} & \frac{-\sqrt{10+2\sqrt{5}}}{4} \\ \frac{\sqrt{10+2\sqrt{5}}}{4} & \frac{-1+\sqrt{5}}{4} \end{bmatrix}$.

- **3×3 counterpart.**

Finally, let's take a quick glance at how the above picture is carried over to the 3×3 case. Don't get carried away, for the complexity of the formula. Today we take a peek at it. We are going to cross-examine it in the forthcoming lectures.

Inverse of a 3×3 matrix.

Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$. The <u>inverse</u> A^{-1} of A is the following matrix:

$$\boxed{A^{-1} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}^{-1} = \frac{1}{\det A} \operatorname{adj} A,}$$

where

$$\det A = a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1,$$

and

$$\operatorname{adj} A = \begin{bmatrix} +\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} & -\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} & +\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \\ -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} & +\begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} & -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \\ +\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} & -\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} & +\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \end{bmatrix}.$$

A^{-1} exists, provided $\det A \neq 0$.

Exercise 5. Decide whether the inverse A^{-1} of A exists, in each of (1–6) below. If it does, then calculate it.

(1) $A = \begin{bmatrix} 2 & 1 & -2 \\ 5 & -4 & -1 \\ 1 & -3 & 4 \end{bmatrix}$. (2) $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 1 \\ 1 & -2 & -2 \end{bmatrix}$.

(3) $A = \begin{bmatrix} 3 & 4 & -4 \\ 2 & 1 & 4 \\ -2 & 4 & 1 \end{bmatrix}$. (4) $A = \begin{bmatrix} 3 & 5 & 10 \\ 3 & 1 & 6 \\ -2 & -2 & -6 \end{bmatrix}$.

(5) $$A = \begin{bmatrix} 1 & \sqrt{2} & \sqrt{3} \\ \sqrt{2} & \dfrac{-2-3\sqrt{6}}{5} & \dfrac{6-\sqrt{6}}{5} \\ \sqrt{3} & \dfrac{6-\sqrt{6}}{5} & \dfrac{-3-2\sqrt{6}}{5} \end{bmatrix}.$$

(6) $$A = \begin{bmatrix} \dfrac{2+3\sqrt{2}}{8} & \dfrac{-2\sqrt{3}+\sqrt{6}}{8} & \dfrac{\sqrt{6}}{4} \\ \dfrac{-2\sqrt{3}+\sqrt{6}}{8} & \dfrac{6+\sqrt{2}}{8} & \dfrac{\sqrt{2}}{4} \\ -\dfrac{\sqrt{6}}{4} & -\dfrac{\sqrt{2}}{4} & \dfrac{\sqrt{2}}{2} \end{bmatrix}.$$