Math 290 ELEMENTARY LINEAR ALGEBRA
REVIEW OF LECTURES – IV

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§4. Matrix arithmetic — II. Multiplications.

• With the knowledge you already have, you can solve a system of linear equations, of 2×2 type.

Example 1. Let's solve

$$\begin{cases} 2x - \ \ y = \ \ 3, \\ 6x + 7y = -5, \end{cases}$$

using the matrix trick. Here we go.

Step 1. Rewrite the system as

$$\underbrace{\begin{bmatrix} 2 & -1 \\ 6 & 7 \end{bmatrix}}_{\substack{\| \\ A}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\substack{\| \\ \boldsymbol{x}}} = \underbrace{\begin{bmatrix} 3 \\ -5 \end{bmatrix}}_{\substack{\| \\ \boldsymbol{b}}}$$

So the equation is of the form $\boxed{A\boldsymbol{x} = \boldsymbol{b}.}$

Remember the golden rule:

$$\boxed{A\boldsymbol{x} = \boldsymbol{b} \quad \underset{\substack{\text{can solve,} \\ \text{if } \det A \neq 0}}{\Longrightarrow} \quad \boldsymbol{x} = A^{-1}\boldsymbol{b}.}$$

Step 2. Find A^{-1}, as follows:

Step 2a. First calculate the determinant of $A = \begin{bmatrix} 2 & -1 \\ 6 & 7 \end{bmatrix}$:

$$\det A = \begin{vmatrix} 2 & -1 \\ 6 & 7 \end{vmatrix} = 2 \cdot 7 - \big(-1\big) \cdot 6$$

$$= 20.$$

Step 2b. Next, form the adjoint of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 6 & 7 \end{bmatrix}$:

$$\operatorname{adj} A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ -6 & 2 \end{bmatrix}.$$

Step 2c. Use the results of Step 2a–b to construct A^{-1}:

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

$$= \frac{1}{20} \begin{bmatrix} 7 & 1 \\ -6 & 2 \end{bmatrix}.$$

Step 3 (Final step) (will have another look momentarily)**.**

Use A^{-1} to find $A^{-1}\boldsymbol{b}$:

$$A^{-1}\boldsymbol{b} = \frac{1}{20} \begin{bmatrix} 7 & 1 \\ -6 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$$= \frac{1}{20} \begin{bmatrix} 7 \cdot 3 + 1 \cdot (-5) \\ (-6) \cdot 3 + 2 \cdot (-5) \end{bmatrix}$$

$$= \frac{1}{20} \begin{bmatrix} 16 \\ -28 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ -\frac{7}{5} \end{bmatrix}.$$

This is the answer $\boldsymbol{x}$. In sum:

$$\underbrace{\begin{bmatrix} 2 & -1 \\ 6 & 7 \end{bmatrix}}_{\substack{\| \\ A}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\substack{\| \\ \boldsymbol{x}}} = \underbrace{\begin{bmatrix} 3 \\ -5 \end{bmatrix}}_{\substack{\| \\ \boldsymbol{b}}} \quad \underset{\text{solve}}{\Longrightarrow} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \boldsymbol{x} = A^{-1}\boldsymbol{b} = \begin{bmatrix} \frac{4}{5} \\ -\frac{7}{5} \end{bmatrix}.$$

- Below let me give an abridged version of the solution:

Problem (same as above). Solve

$$\begin{cases} 2x - \;\; y = \;\; 3, \\ 6x + 7y = -5. \end{cases}$$

Solution. Let

$$A = \begin{bmatrix} 2 & -1 \\ 6 & 7 \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 3 \\ -5 \end{bmatrix},$$

so the given system is $A\boldsymbol{x} = \boldsymbol{b}$. We may solve this as

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = \overbrace{\underbrace{\frac{1}{2 \cdot 7 - \left(-1\right) \cdot 6}}_{\substack{\| \\ \frac{1}{\det A}}} \underbrace{\begin{bmatrix} 7 & 1 \\ -6 & 2 \end{bmatrix}}_{\substack{\| \\ \operatorname{adj} A}}}^{\substack{A^{-1} \\ \|}} \overbrace{\begin{bmatrix} 3 \\ -5 \end{bmatrix}}^{\substack{\boldsymbol{b} \\ \|}}$$

$$= \frac{1}{20} \begin{bmatrix} 7 \cdot 3 + 1 \cdot \left(-5\right) \\ \left(-6\right) \cdot 3 + 2 \cdot \left(-5\right) \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ -\frac{7}{5} \end{bmatrix}.$$

The above is a $\underline{\underline{\text{template}}}$. I want you to write up your solution this way.

• All that said, the last step warrants another look. We 'instinctively' converted the part

$$\begin{bmatrix} 7 & 1 \\ -6 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

into

$$\begin{bmatrix} 7 \cdot 3 \,+\, 1 \cdot (-5) \\ (-6) \cdot 3 \,+\, 2 \cdot (-5) \end{bmatrix}.$$

And that's the correct step. Actually this conversion is something we already 'subconsciously' knew. Indeed, at the very beginning we saw

$$(@) \qquad \begin{cases} 2x - y = 3, \\ 6x + 7y = -5 \end{cases}$$

(the original problem), and we immediately rewrote it as

$$\begin{bmatrix} 2 & -1 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix},$$

which means that we knew that $\begin{bmatrix} 2\,x + (-1)\,y \\ 6\,x + 7\,y \end{bmatrix}$ (= the vector made out of the left-hand sides of the two equations in (@)) and $\begin{bmatrix} 2 & -1 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ are one and the same. Stated in other words, we knew that the correct conversion of $\begin{bmatrix} 2 & -1 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ is $\begin{bmatrix} 2\,x + (-1)\,y \\ 6\,x + 7\,y \end{bmatrix}$. Repeat:

$$\begin{bmatrix} 2 & -1 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2\,x + (-1)\,y \\ 6\,x + \quad 7 \quad y \end{bmatrix}.$$

The following is in a similar vein:

$$\begin{bmatrix} 7 & 1 \\ -6 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 7 \cdot 3 + 1 \cdot (-5) \\ (-6) \cdot 3 + 2 \cdot (-5) \end{bmatrix}.$$

- More generally:

"$\underline{\underline{\textit{The correct conversion of}}} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p \\ r \end{bmatrix} \quad \underline{\underline{\textit{is}}} \quad \begin{bmatrix} ap + br \\ cp + dr \end{bmatrix}.$"

Like last time, we must *officially* declare it to be the rule that is going to be enforced throughout. So, here we go:

- **Rule.**

$$\boxed{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p \\ r \end{bmatrix} = \begin{bmatrix} ap + br \\ cp + dr \end{bmatrix}.}$$

Paraphrase:

$$\boxed{\begin{array}{ll} A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} p \\ r \end{bmatrix} & \\ \Longrightarrow \qquad A\boldsymbol{x} = \begin{bmatrix} ap + br \\ cp + dr \end{bmatrix}. & \end{array}}$$

- I'm sure you got this. But just in case, I want to offer the following breakdown:

Break-down. We are going to do

$$\begin{bmatrix} \boxed{a \quad b} \\ \boxed{c \quad d} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} p \\ r \end{matrix}} \end{bmatrix} = \begin{bmatrix} \boxed{\quad \diamondsuit \quad} \\ \boxed{\quad \clubsuit \quad} \end{bmatrix}.$$

(i) To find $\diamondsuit$, observe

$$\begin{bmatrix} \boxed{a \quad b} \\ c \quad d \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} p \\ r \end{matrix}} \end{bmatrix} = \begin{bmatrix} \boxed{ap + br} \\ \boxed{\clubsuit} \end{bmatrix}.$$

(ii) Next, to find $\clubsuit$, observe

$$\begin{bmatrix} a \quad b \\ \boxed{c \quad d} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} p \\ r \end{matrix}} \end{bmatrix} = \begin{bmatrix} \boxed{ap + br} \\ \boxed{cp + dr} \end{bmatrix}.$$

Example 2. For $A = \begin{bmatrix} 5 & -2 \\ 8 & -9 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$, we have

$$\begin{aligned} A\boldsymbol{x} &= \begin{bmatrix} 5 & -2 \\ 8 & -9 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 5 \cdot 6 + (-2) \cdot 4 \\ 8 \cdot 6 + (-9) \cdot 4 \end{bmatrix} = \begin{bmatrix} 22 \\ 12 \end{bmatrix}. \end{aligned}$$

Exercise 1. Perform each of the following multiplications:

(1) $\begin{bmatrix} 3 & \frac{1}{2} \\ \frac{5}{2} & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. (2) $A\boldsymbol{x}$, where $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} p \\ q \end{bmatrix}$.

(3) $A\boldsymbol{x}$, where $A = \begin{bmatrix} 1 & 2 \\ -6 & 8 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(4) $A\boldsymbol{x}$, where $A = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Exercise 2. Solve each of the folowing systems of equations using matrices:

(1) $\begin{cases} 3x + 6y = 4, \\ 7x + y = 1. \end{cases}$ (2) $\begin{cases} \frac{1}{3}x + 4y = 4, \\ -\frac{2}{3}x + y = \frac{4}{3}. \end{cases}$

- **Matrix multiplication.**

Now let's forget about solving systems of equations. The second topic of the day is completely something else. Well, that's not entirely true — it is actually a tweak of what you've just seen. Instead of multiplying a <u>matrix</u> with a <u>vector</u>, like

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p \\ r \end{bmatrix},$$

how about multiplying a <u>matrix</u> with a <u>matrix</u>, like

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} ?$$

Sure. Here is the rule that we hereby *officially* declare to permanently enforce:

- **Rule.** $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix}.$$

- Paraphrase:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

$$\Longrightarrow \qquad AB = \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix}.$$

Do you clearly see how it works? The following breakdown helps:

- **Break-down:** First and foremost, acknowledge the following:

A and B are both 2×2 matrices $\Longrightarrow$ AB is a 2×2 matrix.

In other words:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} \boxed{} & \boxed{} \\ \boxed{} & \boxed{} \end{bmatrix}.$$

(i) Let us find $\diamondsuit$ in

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} \boxed{\diamondsuit} & \boxed{} \\ \boxed{} & \boxed{} \end{bmatrix}.$$

Since $\diamondsuit$ is in the top-left, accordingly highlight the portion of A and B, like

$$\begin{bmatrix} \boxed{a \quad b} \\ c \quad d \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} p \\ r \end{matrix}} & \begin{matrix} q \\ s \end{matrix} \end{bmatrix} = \begin{bmatrix} \boxed{\diamondsuit} & \boxed{} \\ \boxed{} & \boxed{} \end{bmatrix}.$$

$\diamondsuit$ is $ap + br$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} \boxed{ap+br} & \boxed{} \\ \boxed{} & \boxed{} \end{bmatrix}.$$

(ii) Next, let us find $\heartsuit$ in

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} \boxed{ap+br} & \boxed{\heartsuit} \\ \boxed{} & \boxed{} \end{bmatrix}.$$

Since $\heartsuit$ is in the top-right, accordingly highlight the portion of A and B, like

$$\begin{bmatrix} \boxed{a \quad b} \\ c \quad d \end{bmatrix} \begin{bmatrix} \begin{matrix} p \\ r \end{matrix} & \boxed{\begin{matrix} q \\ s \end{matrix}} \end{bmatrix} = \begin{bmatrix} \boxed{ap+br} & \boxed{\heartsuit} \\ \boxed{} & \boxed{} \end{bmatrix}.$$

$\heartsuit$ is $aq + bs$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} \boxed{ap+br} & \boxed{aq+bs} \\ \boxed{} & \boxed{} \end{bmatrix}.$$

(iii) Similarly, we can find $\clubsuit$ in

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} \boxed{ap+br} & \boxed{aq+bs} \\ \boxed{\clubsuit} & \boxed{} \end{bmatrix}$$

by highlighting

$$\begin{bmatrix} a & b \\ \boxed{c \quad d} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} p \\ r \end{matrix}} & \begin{matrix} q \\ s \end{matrix} \end{bmatrix} = \begin{bmatrix} \boxed{ap+br} & \boxed{aq+bs} \\ \boxed{\clubsuit} & \boxed{} \end{bmatrix}.$$

$\clubsuit$ is $cp + dr$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} \boxed{ap+br} & \boxed{aq+bs} \\ \boxed{cp+dr} & \boxed{} \end{bmatrix}.$$

(iv) Finally, we can find $\spadesuit$ in

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} \boxed{ap+br} & \boxed{aq+bs} \\ \boxed{cp+dr} & \boxed{\spadesuit} \end{bmatrix}$$

by highlighting

$$\begin{bmatrix} a & b \\ \boxed{c \quad d} \end{bmatrix} \begin{bmatrix} \begin{matrix} p \\ r \end{matrix} & \boxed{\begin{matrix} q \\ s \end{matrix}} \end{bmatrix} = \begin{bmatrix} \boxed{ap+br} & \boxed{aq+bs} \\ \boxed{cp+dr} & \boxed{\spadesuit} \end{bmatrix}.$$

$\spadesuit$ is $cq + ds$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} \boxed{ap+br} & \boxed{aq+bs} \\ \boxed{cp+dr} & \boxed{cq+ds} \end{bmatrix}.$$

- In sum, calculating $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ takes four steps :

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} \boxed{\diamondsuit} & \boxed{\heartsuit} \\ \boxed{\clubsuit} & \boxed{\spadesuit} \end{bmatrix}.$$

Those four steps : $\diamondsuit$, $\heartsuit$, $\clubsuit$ and $\spadesuit$, are performed independently.

- **Alternative perspective.** Below is another way to look at it.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

is like

$$\underbrace{\begin{bmatrix} \boxed{\begin{matrix} a & b \\ c & d \end{matrix}} \end{bmatrix}}_{\substack{\parallel \\ A}} \begin{bmatrix} \underset{\substack{\parallel \\ \boldsymbol{x}}}{\boxed{\begin{matrix} p \\ r \end{matrix}}} & \underset{\substack{\parallel \\ \boldsymbol{y}}}{\boxed{\begin{matrix} q \\ s \end{matrix}}} \end{bmatrix},$$

which is basically

$$A \begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \end{bmatrix}.$$

And this is going to be converted to

$$\begin{bmatrix} A\boldsymbol{x} & A\boldsymbol{y} \end{bmatrix},$$

where $A\boldsymbol{x}$ and $A\boldsymbol{y}$ are exactly as we defined earlier.

- **Paraphrase of 'Rule' on page 7.**

$$\boxed{A \begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} A\boldsymbol{x} & A\boldsymbol{y} \end{bmatrix}.}$$

Example 3. For $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix}$, we have

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 2 + 2 \cdot (-1) & 1 \cdot (-1) + 2 \cdot 8 \\ 4 \cdot 2 + 2 \cdot (-1) & 4 \cdot (-1) + 2 \cdot 8 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 15 \\ 6 & 12 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} BA &= \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot 1 + (-1) \cdot 4 & 2 \cdot 2 + (-1) \cdot 2 \\ (-1) \cdot 1 + 8 \cdot 4 & (-1) \cdot 2 + 8 \cdot 2 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 2 \\ 31 & 14 \end{bmatrix}. \end{aligned}$$

- **Important (!)** As this example shows, <u>AB and BA are usually not equal</u>.

Exercise 3. Perform each of the following multiplications:

(1) $\begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 6 & 5 \end{bmatrix}$. (2) $\begin{bmatrix} 1 & -2 \\ -4 & 8 \end{bmatrix} \begin{bmatrix} 3 & 7 \\ -1 & 0 \end{bmatrix}$.

(3) $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & 1 \\ 1 & \frac{-3}{2} \end{bmatrix}$. (4) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(5) AB, where $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, $B = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$,

(6) AB, where $A = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $B = \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

(7) AB, where $A = B = \begin{bmatrix} \frac{-1+\sqrt{5}}{4} & \frac{-\sqrt{10+2\sqrt{5}}}{4} \\ \frac{\sqrt{10+2\sqrt{5}}}{4} & \frac{-1+\sqrt{5}}{4} \end{bmatrix}$.