Math 290 ELEMENTARY LINEAR ALGEBRA
STUDY GUIDE FOR MIDTERM EXAM – A

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§1. What is linear algebra? Overview.

○ We use matrices to rewrite systems of linear equations. Examples:

$$\boxed{\left\{\begin{aligned} 4x + 3y &= 5, \\ 2x - 6y &= -7 \end{aligned}\right.} \quad \underset{\text{"equivalent"}}{\Longleftrightarrow} \quad \boxed{\begin{bmatrix} 4 & 3 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -7 \end{bmatrix}}$$

$$\boxed{\left\{\begin{aligned} 2x_1 - x_2 + 4x_3 &= 1, \\ x_1 + 2x_2 + 5x_3 &= 2, \\ 3x_1 - x_2 + 2x_3 &= 4 \end{aligned}\right.} \quad \underset{\text{"equivalent"}}{\Longleftrightarrow} \quad \boxed{\begin{bmatrix} 2 & -1 & 4 \\ 1 & 2 & 5 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}}$$

○ More on this later. Putting that aside, acknowledge

Formula. The system of equations

$$(\#) \qquad \left\{\begin{aligned} ax + by &= e, \\ cx + dy &= f \end{aligned}\right.$$

is solved as

$$\Big(x,\ y\Big) = \left(\frac{-bf + de}{ad - bc},\ \frac{af - ce}{ad - bc}\right),$$

provided

$$ad - bc \neq 0.$$

The common denominator of the two fractions $ad - bc$ is the determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. So

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Formula. The system of equations

$$(\#\#) \qquad \begin{cases} a_1x_1 + a_2x_2 + a_3x_3 = p, \\ b_1x_1 + b_2x_2 + b_3x_3 = q, \\ c_1x_1 + c_2x_2 + c_3x_3 = r \end{cases}$$

is solved as

$$\Big(x_1,\ x_2,\ x_3\Big) = \Bigg(\frac{p\ b_2c_3 - p\ b_3c_2 - a_2\,q\,c_3 + a_2b_3\ r\ + a_3\,q\,c_2 - a_3b_2\ r}{a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1},$$

$$\frac{a_1\ q\ c_3 - a_1b_3\ r\ - p\ b_1c_3 + p\ b_3c_1 + a_3b_1\ r\ - a_3\ q\ c_1}{a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1},$$

$$\frac{a_1b_2\ r\ - a_1\ q\ c_2 - a_2b_1\ r\ + a_2\ q\ c_1 + p\ b_1c_2 - p\ b_2c_1}{a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1}\Bigg),$$

provided

$$a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 \neq 0.$$

The common denominator of the three fractions is the determinant $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

So

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1.$$

- The determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ naturally popped up out of

$$\begin{cases} ax + by = e, \\ cx + dy = f. \end{cases}$$

Its 3×3 counterpart is the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1,$$

which pops out of

$$\begin{cases} a_1x_1 + a_2x_2 + a_3x_3 = p, \\ b_1x_1 + b_2x_2 + b_3x_3 = q, \\ c_1x_1 + c_2x_2 + c_3x_3 = r. \end{cases}$$

You can effectively use the above formula and solve, for example,

$$\begin{cases} 2x_1 - x_2 + 4x_3 = 1, \\ x_1 + 2x_2 + 5x_3 = 2, \\ 3x_1 - x_2 + 2x_3 = 4, \end{cases}$$

by way of just throwing

$$\begin{array}{lll} a_1 = 2, & a_2 = -1, & a_3 = 4, \\ b_1 = 1, & b_2 = 2, & b_3 = 5, \\ c_1 = 3, & c_2 = -1, & c_3 = 2, \\ p = 1, & q = 2 \ \text{ and } & r = 4 \end{array}$$

into

$$\left(x_1,\ \ x_2,\ \ x_3\right) = \left(\frac{p\ b_2 c_3 - p\ b_3 c_2 - a_2\ q\ c_3 + a_2 b_3\ r\ + a_3\ q\ c_2 - a_3 b_2\ r}{a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1}, \right.$$

$$\frac{a_1\ q\ c_3 - a_1 b_3\ r\ - p\ b_1 c_3 + p\ b_3 c_1 + a_3 b_1\ r\ - a_3\ q\ c_1}{a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1},$$

$$\left. \frac{a_1 b_2\ r\ - a_1\ q\ c_2 - a_2 b_1\ r\ + a_2\ q\ c_1 + p\ b_1 c_2 - p\ b_2 c_1}{a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1} \right).$$

That way you readily get the answer

$$\left(x_1,\ \ x_2,\ \ x_3\right) = \left(\frac{47}{23},\ \frac{27}{23},\ -\frac{11}{23}\right).$$

Here, a part of the calculation is $a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1$,

which is the determinant $\begin{vmatrix} 2 & -1 & 4 \\ 1 & 2 & 5 \\ 3 & -1 & 2 \end{vmatrix}$:

$$\begin{aligned}
&\begin{vmatrix} 2 & -1 & 4 \\ 1 & 2 & 5 \\ 3 & -1 & 2 \end{vmatrix} \\
&= 2 \cdot 2 \cdot 2 - 2 \cdot 5 \cdot \left(-1\right) - \left(-1\right) \cdot 1 \cdot 2 + \left(-1\right) \cdot 5 \cdot 3 + 4 \cdot 1 \cdot \left(-1\right) - 4 \cdot 2 \cdot 3 \\
&= 8 - \left(-10\right) - \left(-2\right) + \left(-15\right) + \left(-4\right) - 24 \\
&= -23.
\end{aligned}$$

Question 1. In the above example, the calculation was quite involved due to the complexity of the formula. Is there another, simpler formula?

— The short answer is ‘no’.

Question 2. If that’s the case, is there a method that algorithmically deduces the answer, for each concrete system of equations $\left(3 \times 3\right)$, something like the above, without ever relying on the formula?

— The short answer is ‘yes’: <u>Gaussian elimination method</u> $\big($later$\big)$.

- Just take a look:

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

$$\begin{aligned} = \quad & a_1b_2c_3d_4 - a_1b_2c_4d_3 - a_1b_3c_2d_4 + a_1b_3c_4d_2 + a_1b_4c_2d_3 - a_1b_4c_3d_2 \\ & -a_2b_1c_3d_4 + a_2b_1c_4d_3 + a_2b_3c_1d_4 - a_2b_3c_4d_1 - a_2b_4c_1d_3 + a_2b_4c_3d_1 \\ & +a_3b_1c_2d_4 - a_3b_1c_4d_2 - a_3b_2c_1d_4 + a_3b_2c_4d_1 + a_3b_4c_1d_2 - a_3b_4c_2d_1 \\ & -a_4b_1c_2d_3 + a_4b_1c_3d_2 + a_4b_2c_1d_3 - a_4b_2c_3d_1 - a_4b_3c_1d_2 + a_4b_3c_2d_1 \ . \end{aligned}$$

§2. Determinants – Intro.

Definition (Determinant; 2 × 2).

The determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is defined as follows:

$$\boxed{\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.}$$

— So, for example:

Example.

$$\begin{vmatrix} 7 & 5 \\ 2 & 1 \end{vmatrix} = 7 \cdot 1 - 5 \cdot 2$$

$$= -3.$$

Not that hard. However, there is something I want to stress:

Determinant is defined for each matrix,

meaning:

$$\boxed{\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad \underline{\underline{\text{is the determinant of the matrix}}} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}.}$$

For example, $\begin{vmatrix} 7 & 5 \\ 2 & 1 \end{vmatrix} = -3$ (as we have just calculated) is regarded as the determinant of the matrix $\begin{bmatrix} 7 & 5 \\ 2 & 1 \end{bmatrix}$. By implication: For your successful grasp of the concept of determinants, you need to agree on the following first and foremost:

- *First there is this notion of matrices.*
- *Then the determinant is defined for each matrix.*
- *Matrices themselves are arrays, whereas:*
- *The determinant of a matrix is a scalar.*

- We use a letter, typically a capital letter, to represent a matrix. So we say

$$\text{“Let } A \text{ stand for the matrix } \begin{bmatrix} a & b \\ c & d \end{bmatrix}.\text{”}$$

Or just

$$\text{“Let} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.\text{”}$$

Taking all these intoa account:

Official Definition of Determinant (2 × 2).

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Define <u>the determinant of the matrix A</u> as

$$\boxed{\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.}$$

Example. (1) For $A = \begin{bmatrix} -6 & 2 \\ 8 & -4 \end{bmatrix}$, its determinant is

$$\det A = \begin{vmatrix} -6 & 2 \\ 8 & -4 \end{vmatrix} = (-6)\cdot(-4) - 2\cdot 8$$

$$= 8.$$

(2) For $A = \begin{bmatrix} -2 & 4 \\ -3 & 6 \end{bmatrix}$, its determinant is

$$\det A = \begin{vmatrix} -2 & 4 \\ -3 & 6 \end{vmatrix} = (-2)\cdot 6 - 4\cdot(-3)$$

$$= 0.$$

(3) For $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, its determinant is

$$\det A = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1\cdot 1 - 0\cdot 0$$

$$= 1.$$

Exercise (= "II"; Exercise 1). Calculate:

(1) $\begin{vmatrix} 1 & 6 \\ 1 & 3 \end{vmatrix}$. (2) $\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}$. (3) $\begin{vmatrix} 2 & 5 \\ \frac{3}{10} & 4 \end{vmatrix}$.

(4) $\det A$, where $A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}$.

(5a) $\det A$, where $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. (5b) $\det B$, where $B = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

(6a) $\det A$, where $A = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$. (6b) $\det B$, where $B = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$.

(7a) $\det A$, where $A = \begin{bmatrix} -1+\sqrt{5} & -\sqrt{10+2\sqrt{5}} \\ \sqrt{10+2\sqrt{5}} & -1+\sqrt{5} \end{bmatrix}$.

(7b) $\det B$, where $B = \begin{bmatrix} \frac{-1+\sqrt{5}}{4} & \frac{-\sqrt{10+2\sqrt{5}}}{4} \\ \frac{\sqrt{10+2\sqrt{5}}}{4} & \frac{-1+\sqrt{5}}{4} \end{bmatrix}$.

- Now, keeping the narrative intact, let's define the 3×3 determinant:

Official Definition of Determinant (3×3).

Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$. Define <u>the determinant of the matrix A</u> as

$$\det A = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1.$$

- **Co-factoring.**

Observe

$$\begin{aligned} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 \\ &= a_1 \Big(b_2 c_3 - b_3 c_2\Big) - a_2 \Big(b_1 c_3 - b_3 c_1\Big) + a_3 \Big(b_1 c_2 - b_2 c_1\Big) \\ &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}. \end{aligned}$$

More generally:

$$(\mathrm{i})_a \qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

$$(\mathrm{i})_b \qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}.$$

$$(\mathrm{i})_c \qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

$$(\mathrm{ii})_1 \qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}.$$

$$(\mathrm{ii})_2 \qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}.$$

$$(\mathrm{ii})_3 \qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

Exercise (= "II"; Exercise 2). Compare these six lines and discern their patterns (including the signs that come with the terms).

These show that a 3×3 determinant is formed through 2×2 determinants. This is a part of the bigger picture: There is a hierarchical structure existing among the expressions of different size determinants $\left(2 \times 2;\ 3 \times 3;\ 4 \times 4;\ \cdots\right)$.

Example. For $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -2 \\ 3 & -1 & 4 \end{bmatrix}$, calculate $\det A = \begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & -2 \\ 3 & -1 & 4 \end{vmatrix}$.

We may directly apply the definition of the determinant:

$$\det A = \begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & -2 \\ 3 & -1 & 4 \end{vmatrix}$$

$$= 1 \cdot 1 \cdot 4 - 1 \cdot \left(-2\right) \cdot \left(-1\right) - 2 \cdot 0 \cdot 4 + 2 \cdot \left(-2\right) \cdot 3 + 2 \cdot 0 \cdot \left(-1\right) - 2 \cdot 1 \cdot 3$$

$$= 4 - 2 - 0 + \left(-12\right) + 0 - 6 = -16.$$

But we could've applied the co-factoring, say $(\mathrm{i})_a$, for the same problem instead:

$$\begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & -2 \\ 3 & -1 & 4 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & -2 \\ -1 & 4 \end{vmatrix} - 2 \cdot \begin{vmatrix} 0 & -2 \\ 3 & 4 \end{vmatrix} + 2 \cdot \begin{vmatrix} 0 & 1 \\ 3 & -1 \end{vmatrix}$$

$$= 1 \cdot 2 - 2 \cdot 6 + 2 \cdot \left(-3\right) = -16.$$

Or, we could've applied a different co-factoring, say $(\mathrm{ii})_2$ instead:

$$\begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & -2 \\ 3 & -1 & 4 \end{vmatrix} = -2 \cdot \begin{vmatrix} 0 & -2 \\ 3 & 4 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} - \left(-1\right) \cdot \begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix}$$

$$= -2 \cdot 6 + 1 \cdot \left(-2\right) - \left(-1\right) \cdot \left(-2\right) = -16.$$

Exercise (= "II"; Exercise 3). Calculate:

(1) $\begin{vmatrix} 5 & 6 & 1 \\ 1 & 3 & -4 \\ 2 & 5 & 2 \end{vmatrix}$. (2) $\begin{vmatrix} 1 & -1 & 0 \\ 3 & 1 & 1 \\ -2 & 2 & 2 \end{vmatrix}$. (3) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$.

(4) $\det A$, where $A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$.

(5) $\det A$, where $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$.

Factor the answer for (5).

(6) $\det A$, where $A = \begin{bmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{bmatrix}$.

(7) $\det A$, where $A = \begin{bmatrix} 1 & x & x^2 \\ x & 1 & x^3 \\ x^2 & x^3 & 1 \end{bmatrix}$.

Factor the answer for (7).

(8)* $\det A$, where

$$A = \begin{bmatrix} a^2-b^2-c^2+d^2 & 2\Big(ab+cd\Big) & 2\Big(-ac+bd\Big) \\ 2\Big(-ab+cd\Big) & a^2-b^2+c^2-d^2 & 2\Big(ad+bc\Big) \\ 2\Big(ac+bd\Big) & 2\Big(-ad+bc\Big) & a^2+b^2-c^2-d^2 \end{bmatrix}.$$

Factor the answer for (8).

§3. Matrix arithmetic — I. Inverse of a matrix.

- Recall from §1:

$$(*)\quad \boxed{\begin{cases} 4x + 3y = 5, \\ 2x - 6y = -7 \end{cases}} \quad \underset{\text{“equivalent”}}{\Longleftrightarrow} \quad \boxed{\begin{bmatrix} 4 & 3 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -7 \end{bmatrix}}$$

The box on the right is a paraphrase of the box on the left. The right box:

$$\underbrace{\begin{bmatrix} 4 & 3 \\ 2 & -6 \end{bmatrix}}_{\substack{\| \\ A}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\substack{\| \\ \boldsymbol{x}}} = \underbrace{\begin{bmatrix} 5 \\ -7 \end{bmatrix}}_{\substack{\| \\ \boldsymbol{b}}}$$

is basically

$$\boxed{A\boldsymbol{x} = \boldsymbol{b}.}$$

Just like the equation $ax = b$, where all the letters in sight are scalars, is solved as $x = a^{-1}b$, we want to solve the above as

$$\text{“}\ \boldsymbol{x} = A^{-1}\boldsymbol{b}\ \text{”}.$$

Good news:

> $\underline{\underline{A^{-1} \textit{ makes sense, as a } 2 \times 2 \textit{ matrix, and thus}}}$ $A^{-1}\boldsymbol{b}$ $\underline{\underline{\textit{also makes sense,}}}$
>
> $\underline{\underline{\textit{under one condition}}}$: $\det A \neq 0$.

$$\boxed{A\boldsymbol{x} = \boldsymbol{b} \quad \underset{\substack{\text{can solve,} \\ \text{if } \det A \neq 0}}{\Longrightarrow} \quad \boldsymbol{x} = A^{-1}\boldsymbol{b}.}$$

This is a legit way to solve the equation $A\boldsymbol{x} = \boldsymbol{b}$. Here, most importantly, A^{-1} is below:

Inverse of a 2×2 matrix.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The <u>inverse</u> A^{-1} of A is the following matrix:

$$\boxed{A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}}.$$

A^{-1} exists, provided $\det A = ad - bc \neq 0$.

We would much rather write it like

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

This is acceptable, with the proviso we adopt the following definition:

- **Definition (Scalar multiplied to a matrix).** Let s be a scalar. Then

$$\boxed{s \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} sa & sb \\ sc & sd \end{bmatrix}.}$$

Paraphrase:

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and s : a scalar

$\Longrightarrow$ $sA = \begin{bmatrix} sa & sb \\ sc & sd \end{bmatrix}$.

Example. (1)
$$3\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} = \begin{bmatrix}3 & 6\\ 9 & 12\end{bmatrix}.$$

(2)
$$4\begin{bmatrix}3 & 3\\ 3 & 3\end{bmatrix} = \begin{bmatrix}12 & 12\\ 12 & 12\end{bmatrix}.$$

(3)
$$\frac{1}{7}\begin{bmatrix}5 & 7\\ -1 & 0\end{bmatrix} = \begin{bmatrix}\frac{5}{7} & 1\\ \frac{-1}{7} & 0\end{bmatrix}.$$

(4)
$$\frac{9}{2}\begin{bmatrix}\frac{2}{9} & 2\\ \frac{4}{9} & \frac{1}{9}\end{bmatrix} = \begin{bmatrix}1 & 9\\ 2 & \frac{1}{2}\end{bmatrix}.$$

(5)
$$1\begin{bmatrix}0 & -2\\ 6 & 3\end{bmatrix} = \begin{bmatrix}0 & -2\\ 6 & 3\end{bmatrix}.$$

- An obvious generalization of (5) is

$$\boxed{1\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}.}$$

Paraphrase:

$$\boxed{\text{If} \quad A = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \quad \Longrightarrow \quad 1A = A.}$$

- **Definition (negation).**

$$\boxed{-\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}-a & -b\\ -c & -d\end{bmatrix}.}$$

Paraphrase:

$$\boxed{\text{If} \quad A = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \quad \Longrightarrow \quad -A = \begin{bmatrix}-a & -b\\ -c & -d\end{bmatrix}.}$$

Example. $$0\begin{bmatrix}1&1\\2&3\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}, \qquad 8\begin{bmatrix}0&0\\0&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}.$$

- An obvious generalization of the above is

$$0\begin{bmatrix}a&b\\c&d\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}, \qquad s\begin{bmatrix}0&0\\0&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}.$$

- We denote $\begin{bmatrix}0&0\\0&0\end{bmatrix}$ as O. Then we can paraphrase it as:

$$\boxed{\begin{array}{lll}\text{If} \quad A=\begin{bmatrix}a&b\\c&d\end{bmatrix} & \text{and} & s : \text{ a scalar}\\ \Longrightarrow & 0A = O, & sO = O.\end{array}}$$

Example. $$\left(-1\right)\begin{bmatrix}3&4\\5&9\end{bmatrix}=\begin{bmatrix}-3&-4\\-5&-9\end{bmatrix}.$$

$$-\begin{bmatrix}3&4\\5&9\end{bmatrix}=\begin{bmatrix}-3&-4\\-5&-9\end{bmatrix}.$$

- As you can clearly see,

$$\boxed{\left(-1\right)\begin{bmatrix}a&b\\c&d\end{bmatrix}=-\begin{bmatrix}a&b\\c&d\end{bmatrix}.}$$

Paraphrase:

$$\boxed{\text{If} \quad A=\begin{bmatrix}a&b\\c&d\end{bmatrix} \quad \Longrightarrow \quad \left(-1\right)A = -A.}$$

Exercise (= "III"; Exercise 1). Write each of the following in the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

(1) $3\begin{bmatrix} -4 & 2 \\ 6 & 5 \end{bmatrix}$. (2) $\dfrac{1}{2}\begin{bmatrix} 10 & 12 \\ 8 & 4 \end{bmatrix}$. (3) $\dfrac{1}{8}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(4) $\left(-2\right)\begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}$. (5) $1\begin{bmatrix} 7 & -5 \\ \frac{1}{2} & 1 \end{bmatrix}$. (6) $0\begin{bmatrix} 124 & 242 \\ 163 & 89 \end{bmatrix}$.

(7) $1000\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Exercise (= "III"; Exercise 2). Write each of the following in the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

(1) $-\begin{bmatrix} -6 & -8 \\ 3 & 4 \end{bmatrix}$. (2) $-\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. (3) $-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Exercise (= "III"; Exercise 3). For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, define

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \qquad \Big(\text{the \underline{ transpose } of } A\Big).$$

Assume $A^T = -A$. Prove that there is a scalar s such that

$$A = s\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Inverse of a 2×2 matrix, paraphrased.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The $\underline{\text{ inverse }}$ A^{-1} of A is the following matrix:

$$\boxed{\begin{aligned} A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} &= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \end{aligned}}$$

A^{-1} exists, provided $\det A = ad - bc \neq 0$.

- **Adjoint matrix.**

For convenience of reference, we give it a name for a part of the A^{-1} formation:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Longrightarrow \quad A^{-1} = \frac{1}{\det A} \underbrace{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}_{\substack{\| \\ \operatorname{adj} A}}$$

So,

$$\boxed{A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Longrightarrow \quad \operatorname{adj} A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.}$$

We call $\operatorname{adj} A$ the $\underline{\text{ adjoint matrix }}$ of A.

- We may accordingly further paraphrase the above:

Inverse of a 2×2 matrix, paraphrased — II.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The $\underline{\text{ inverse }}$ A^{-1} of A is the following matrix:

$$\boxed{A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det A} \operatorname{adj} A,}$$

where

$$\det A = ad - bc, \qquad \text{and}$$

$$\operatorname{adj} A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

A^{-1} exists, provided $\det A = ad - bc \neq 0$.

- Let's calculate A^{-1} for some concete matrix A.

Example. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$. Find A^{-1}.

Here is how it goes:

Step 1.

$$\det A = \begin{vmatrix} 2 & 3 \\ 4 & 7 \end{vmatrix} = 2 \cdot 7 - 3 \cdot 4$$

$$= 2.$$

Step 2.

$$\text{adj}\, A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -3 \\ -4 & 2 \end{bmatrix}.$$

Step 3.

$$A^{-1} = \frac{1}{\det A}\, \text{adj}\, A$$

$$= \frac{1}{2} \begin{bmatrix} 7 & -3 \\ -4 & 2 \end{bmatrix}$$

$$\left(= \begin{bmatrix} \frac{7}{2} & \frac{-3}{2} \\ -2 & 1 \end{bmatrix} \right).$$

Example 5. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -5 & 3 \end{bmatrix}$. Find A^{-1}.

Step 1.

$$\det A = \begin{vmatrix} 3 & -2 \\ -5 & 3 \end{vmatrix} = 3 \cdot 3 - \left(-2\right) \cdot \left(-5\right)$$

$$= -1.$$

Step 2.

$$\text{adj}\, A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}.$$

Step 3.

$$A^{-1} = \frac{1}{\det A}\, \text{adj}\, A$$

$$= \frac{1}{-1} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix}.$$

- **What if the determinant of A equals 0 ?**

  When $\det A = 0$, the inverse A^{-1} does not exist.

Example. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 1 & 2 \end{bmatrix}$. Decide whether A^{-1} exists.

For that matter, it suffices to calculate $\det A$:

$$\det A = \begin{vmatrix} 4 & 8 \\ 1 & 2 \end{vmatrix} = 4 \cdot 2 - 8 \cdot 1$$

$$= 0.$$

So, we conclude that A^{-1} does not exist.

Exercise (= "III"; Exercise 4). Decide whether A^{-1} exists, in each of (1–12) below. If it does, then calculate it.

(1) $A = \begin{bmatrix} 5 & 7 \\ -1 & 3 \end{bmatrix}$. (2) $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$. (3) $A = \begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix}$.

(4) $A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$. (5) $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$. (6) $A = \begin{bmatrix} 1 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{9} \end{bmatrix}$.

(7) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. (8) $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. (9) $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(10) $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$. (11) $A = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$.

(12) $A = \begin{bmatrix} \frac{-1+\sqrt{5}}{4} & \frac{-\sqrt{10+2\sqrt{5}}}{4} \\ \frac{\sqrt{10+2\sqrt{5}}}{4} & \frac{-1+\sqrt{5}}{4} \end{bmatrix}$.

Inverse of a 3×3 matrix.

Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$. The <u>inverse</u> A^{-1} of A is the following matrix:

$$\boxed{A^{-1} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}^{-1} = \frac{1}{\det A} \operatorname{adj} A,}$$

where

$$\det A = a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1,$$

and

$$\operatorname{adj} A = \begin{bmatrix} + \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} & - \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} & + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \\ - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} & + \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} & - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \\ + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} & - \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} & + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \end{bmatrix}.$$

A^{-1} exists, provided $\det A \neq 0$.

Exercise (= "III"; Exercise 5). Decide whether A^{-1} exists, in each of (1–6) below. If it does, then calculate it.

$$(1)\quad A = \begin{bmatrix} 2 & 1 & -2 \\ 5 & -4 & -1 \\ 1 & -3 & 4 \end{bmatrix}. \qquad (2)\quad A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 1 \\ 1 & -2 & -2 \end{bmatrix}.$$

$$(3)\quad A = \begin{bmatrix} 3 & 4 & -4 \\ 2 & 1 & 4 \\ -2 & 4 & 1 \end{bmatrix}. \qquad (4)\quad A = \begin{bmatrix} 3 & 5 & 10 \\ 3 & 1 & 6 \\ -2 & -2 & -6 \end{bmatrix}.$$

$$(5)\quad A = \begin{bmatrix} 1 & \sqrt{2} & \sqrt{3} \\ \sqrt{2} & \dfrac{-2-3\sqrt{6}}{5} & \dfrac{6-\sqrt{6}}{5} \\ \sqrt{3} & \dfrac{6-\sqrt{6}}{5} & \dfrac{-3-2\sqrt{6}}{5} \end{bmatrix}.$$

$$(6)\quad A = \begin{bmatrix} \dfrac{2+3\sqrt{2}}{8} & \dfrac{-2\sqrt{3}+\sqrt{6}}{8} & \dfrac{\sqrt{6}}{4} \\ \dfrac{-2\sqrt{3}+\sqrt{6}}{8} & \dfrac{6+\sqrt{2}}{8} & \dfrac{\sqrt{2}}{4} \\ -\dfrac{\sqrt{6}}{4} & -\dfrac{\sqrt{2}}{4} & \dfrac{\sqrt{2}}{2} \end{bmatrix}.$$

§4. Matrix arithmetic — II. Multiplications.

- We can now solve a system of linear equations, of 2×2 type, using matrices.

Example. Solve

$$\begin{cases} 2x - \ \ y = \ \ 3, \\ 6x + 7y = -5, \end{cases}$$

using the matrix trick.

Solution. Let

$$A = \begin{bmatrix} 2 & -1 \\ 6 & 7 \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 3 \\ -5 \end{bmatrix},$$

so the given system is $A\boldsymbol{x} = \boldsymbol{b}$. We may solve this as

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = \overbrace{\underbrace{\frac{1}{2 \cdot 7 - \left(-1\right) \cdot 6}}_{\substack{\| \\ \frac{1}{\det A}}} \underbrace{\begin{bmatrix} 7 & 1 \\ -6 & 2 \end{bmatrix}}_{\substack{\| \\ \operatorname{adj} A}}}^{\substack{A^{-1} \\ \|}} \overbrace{\begin{bmatrix} 3 \\ -5 \end{bmatrix}}^{\substack{\boldsymbol{b} \\ \|}}$$

$$= \frac{1}{20} \begin{bmatrix} 7 \cdot 3 + 1 \cdot \left(-5\right) \\ \left(-6\right) \cdot 3 + 2 \cdot \left(-5\right) \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ -\frac{7}{5} \end{bmatrix}.$$

- Here, in the last step the right conversion of $\begin{bmatrix} 7 & 1 \\ -6 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ is

$$\begin{bmatrix} 7 \cdot 3 + 1 \cdot \left(-5\right) \\ \left(-6\right) \cdot 3 + 2 \cdot \left(-5\right) \end{bmatrix}.$$

- More generally:

"*The correct conversion of* $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p \\ r \end{bmatrix}$ *is* $\begin{bmatrix} ap + br \\ cp + dr \end{bmatrix}$. "

• **Rule.**

$$\boxed{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p \\ r \end{bmatrix} = \begin{bmatrix} ap + br \\ cp + dr \end{bmatrix}.}$$

Paraphrase:

$$\boxed{\begin{array}{ccc} A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} p \\ r \end{bmatrix} & & \\ \Longrightarrow & & A\boldsymbol{x} = \begin{bmatrix} ap + br \\ cp + dr \end{bmatrix}. \end{array}}$$

Break-down. We are going to do

$$\begin{bmatrix} \boxed{a \quad b} \\ \boxed{c \quad d} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} p \\ r \end{matrix}} \end{bmatrix} = \begin{bmatrix} \boxed{\quad\diamondsuit\quad} \\ \boxed{\quad\clubsuit\quad} \end{bmatrix}.$$

(i) To find $\diamondsuit$, observe

$$\begin{bmatrix} \boxed{a \quad b} \\ c \quad d \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} p \\ r \end{matrix}} \end{bmatrix} = \begin{bmatrix} \boxed{ap + br} \\ \boxed{\quad\clubsuit\quad} \end{bmatrix}.$$

(ii) Next, to find $\clubsuit$, observe

$$\begin{bmatrix} a \quad b \\ \boxed{c \quad d} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} p \\ r \end{matrix}} \end{bmatrix} = \begin{bmatrix} \boxed{ap + br} \\ \boxed{cp + dr} \end{bmatrix}.$$

Example. For $A = \begin{bmatrix} 5 & -2 \\ 8 & -9 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$, we have

$$\begin{aligned} A\boldsymbol{x} &= \begin{bmatrix} 5 & -2 \\ 8 & -9 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 5 \cdot 6 + (-2) \cdot 4 \\ 8 \cdot 6 + (-9) \cdot 4 \end{bmatrix} = \begin{bmatrix} 22 \\ 12 \end{bmatrix}. \end{aligned}$$

Exercise (= "IV"; Exercise 1). Perform each of the following multiplications:

(1) $\begin{bmatrix} 3 & \frac{1}{2} \\ \frac{5}{2} & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. (2) $A\boldsymbol{x}$, where $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} p \\ q \end{bmatrix}$.

(3) $A\boldsymbol{x}$, where $A = \begin{bmatrix} 1 & 2 \\ -6 & 8 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(4) $A\boldsymbol{x}$, where $A = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Exercise (= "IV"; Exercise 2). Solve each of the folowing systems of equations using matrices:

(1) $\begin{cases} 3x + 6y = 4, \\ 7x + \ \ y = 1. \end{cases}$ (2) $\begin{cases} \dfrac{1}{3}x + 4y = 4, \\ -\dfrac{2}{3}x + \ \ y = \dfrac{4}{3}. \end{cases}$

- **Matrix multiplication.** How about multiplying out two matrices, like

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} ?$$

- **Rule.**

$$\boxed{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix}.}$$

- Paraphrase:

$$\boxed{\begin{aligned} & A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \\ \Longrightarrow \quad & AB = \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix}. \end{aligned}}$$

- **Break-down:** First and foremost,

$$A \text{ and } B \text{ are both } 2 \times 2 \text{ matrices} \quad \Longrightarrow \quad AB \text{ is a } 2 \times 2 \text{ matrix.}$$

So

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} \boxed{} & \boxed{} \\ \boxed{} & \boxed{} \end{bmatrix}.$$

(i) We can find $\diamondsuit$ in

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} \boxed{\quad\diamondsuit\quad} & \boxed{} \\ \boxed{} & \boxed{} \end{bmatrix}$$

as

$$\begin{bmatrix} \boxed{a \quad b} \\ c \quad d \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} p \\ r \end{matrix}} & \begin{matrix} q \\ s \end{matrix} \end{bmatrix} = \begin{bmatrix} \boxed{ap + br} & \boxed{} \\ \boxed{} & \boxed{} \end{bmatrix}.$$

(ii) We can find $\heartsuit$ in

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} \boxed{ap + br} & \boxed{\quad\heartsuit\quad} \\ \boxed{} & \boxed{} \end{bmatrix}$$

as

$$\begin{bmatrix} \boxed{a \quad b} \\ c \quad d \end{bmatrix} \begin{bmatrix} p & \\ r & \end{bmatrix} = \begin{bmatrix} \boxed{ap + br} & \boxed{aq + bs} \\ \boxed{} & \boxed{} \end{bmatrix}.$$

(iii) We can find $\clubsuit$ in

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} \boxed{ap + br} & \boxed{aq + bs} \\ \boxed{\quad\clubsuit\quad} & \boxed{} \end{bmatrix}$$

as

$$\begin{bmatrix} a & b \\ \hline \multicolumn{1}{|c}{c} & \multicolumn{1}{c|}{d} \\ \hline \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} p \\ r \end{matrix}} & \begin{matrix} q \\ s \end{matrix} \end{bmatrix} = \begin{bmatrix} \boxed{ap+br} & \boxed{aq+bs} \\ \boxed{cp+dr} & \boxed{} \end{bmatrix}.$$

(iv) Finally, we can find $\spadesuit$ in

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} \boxed{ap+br} & \boxed{aq+bs} \\ \boxed{cp+dr} & \boxed{\spadesuit} \end{bmatrix}$$

as

$$\begin{bmatrix} \begin{matrix} a & b \end{matrix} \\ \boxed{\begin{matrix} c & d \end{matrix}} \end{bmatrix} \begin{bmatrix} \begin{matrix} p \\ r \end{matrix} & \boxed{\begin{matrix} q \\ s \end{matrix}} \end{bmatrix} = \begin{bmatrix} \boxed{ap+br} & \boxed{aq+bs} \\ \boxed{cp+dr} & \boxed{cq+ds} \end{bmatrix}.$$

- **Alternative perspective.** $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ is like

$$\underbrace{\begin{bmatrix} \boxed{\begin{matrix} a & b \\ c & d \end{matrix}} \end{bmatrix}}_{\substack{\| \\ A}} \begin{bmatrix} \underbrace{\boxed{\begin{matrix} p \\ r \end{matrix}}}_{\substack{\| \\ \boldsymbol{x}}} & \underbrace{\boxed{\begin{matrix} q \\ s \end{matrix}}}_{\substack{\| \\ \boldsymbol{y}}} \end{bmatrix},$$

which is basically

$$A \begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \end{bmatrix}.$$

And this is going to be converted to

$$\begin{bmatrix} A\boldsymbol{x} & A\boldsymbol{y} \end{bmatrix}.$$

Example. For $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix}$, we have

$$
\begin{aligned}
AB &= \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix} \\
&= \begin{bmatrix} 1 \cdot 2 + 2 \cdot \big(-1\big) & 1 \cdot \big(-1\big) + 2 \cdot 8 \\ 4 \cdot 2 + 2 \cdot \big(-1\big) & 4 \cdot \big(-1\big) + 2 \cdot 8 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 15 \\ 6 & 12 \end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
BA &= \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 2 \cdot 1 + \big(-1\big) \cdot 4 & 2 \cdot 2 + \big(-1\big) \cdot 2 \\ \big(-1\big) \cdot 1 + 8 \cdot 4 & \big(-1\big) \cdot 2 + 8 \cdot 2 \end{bmatrix} \\
&= \begin{bmatrix} -2 & 2 \\ 31 & 14 \end{bmatrix}.
\end{aligned}
$$

• **Important (!)** As this example shows, $\underline{AB \text{ and } BA \text{ are usually not equal}}$.

Exercise (= "IV"; Exercise 3). Perform each of the following multiplications:

(1) $\begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 6 & 5 \end{bmatrix}$. (2) $\begin{bmatrix} 1 & -2 \\ -4 & 8 \end{bmatrix} \begin{bmatrix} 3 & 7 \\ -1 & 0 \end{bmatrix}$.

(3) $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & 1 \\ 1 & \frac{-3}{2} \end{bmatrix}$. (4) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(5) AB, where $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, $B = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$,

(6) AB, where $A = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $B = \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

(7) AB, where $A = B = \begin{bmatrix} \frac{-1+\sqrt{5}}{4} & \frac{-\sqrt{10+2\sqrt{5}}}{4} \\ \frac{\sqrt{10+2\sqrt{5}}}{4} & \frac{-1+\sqrt{5}}{4} \end{bmatrix}$.

§5. Matrix arithmetic — III. Identity matrix.

Definition. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the (2×2) identity matrix. Write

$$\boxed{I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.}$$

Example. For $A = \begin{bmatrix} 2 & 4 \\ 7 & -3 \end{bmatrix}$, we have

$$\boxed{IA = A, \quad \text{and} \quad AI = A.}$$

Indeed,

$$\begin{aligned} IA &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 7 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 2 + 0 \cdot 7 & 1 \cdot 4 + 0 \cdot (-3) \\ 0 \cdot 2 + 1 \cdot 7 & 0 \cdot 4 + 1 \cdot (-3) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 7 & -3 \end{bmatrix} = A, \end{aligned}$$

$$\begin{aligned} AI &= \begin{bmatrix} 2 & 4 \\ 7 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot 1 + 4 \cdot 0 & 2 \cdot 0 + 4 \cdot 1 \\ 7 \cdot 1 + (-3) \cdot 0 & 7 \cdot 0 + (-3) \cdot 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 7 & -3 \end{bmatrix} = A. \end{aligned}$$

This is not a coincidence:

Fact 1. For $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have

$$\boxed{IA = A, \quad \text{and} \quad AI = A.}$$

- This is something that requires a proof.

Proof. Do IA and AI for

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \text{and} \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

$$\begin{aligned} IA &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot a + 0 \cdot c & 1 \cdot b + 0 \cdot d \\ 0 \cdot a + 1 \cdot c & 0 \cdot b + 1 \cdot d \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A, \end{aligned}$$

$$\begin{aligned} AI &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} a \cdot 1 + b \cdot 0 & a \cdot 0 + b \cdot 1 \\ c \cdot 1 + d \cdot 0 & c \cdot 0 + d \cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A. \qquad \square \end{aligned}$$

Analogy:

"In the context of matrix multiplications, the identity matrix 'I' serves the same role as '1' (the number) does in the usual number multiplications. We always have

$$(*) \qquad 1\,a = a, \quad \text{and} \quad a\,1 = a$$

for any number a. In the same token,

$$(\#) \qquad I\,A = A, \quad \text{and} \quad A\,I = A$$

for any matrix A. These two, $(*)$ and $(\#)$, are entirely parallel."

Quiz. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and suppose A^{-1} exists.

$$AA^{-1} = ? \qquad A^{-1}A = ?$$

Answer. $\boxed{AA^{-1} = I, \quad \text{and} \quad A^{-1}A = I.}$

Proof. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$A^{-1} = \frac{1}{ad - bc}\,\text{adj}\,A, \qquad \text{where}$$

$$\text{adj}\,A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

In that regard, first calculate $A\left(\text{adj}\,A\right)$ and $\left(\text{adj}\,A\right)A$ each:

$$\begin{aligned}
A\left(\operatorname{adj} A\right) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\
&= \begin{bmatrix} ad + b\left(-c\right) & a\left(-b\right) + ba \\ cd + d\left(-c\right) & c\left(-b\right) + da \end{bmatrix} \\
&= \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \\
&= \left(ad - bc\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \left(ad - bc\right) I,
\end{aligned}$$

$$\begin{aligned}
\left(\operatorname{adj} A\right) A &= \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
&= \begin{bmatrix} da + \left(-b\right)c & db + \left(-b\right)d \\ \left(-c\right)a + ac & \left(-c\right)b + ad \end{bmatrix} \\
&= \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \\
&= \left(ad - bc\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \left(ad - bc\right) I.
\end{aligned}$$

In short,

$$A\left(\operatorname{adj} A\right) = \left(ad - bc\right) I, \qquad \text{and} \qquad \left(\operatorname{adj} A\right) A = \left(ad - bc\right) I.$$

Now, suppose $ad - bc \neq 0$. Then you can divide the two sides of each of the above two equalities:

$$A \underbrace{\left(\frac{1}{ad - bc} \operatorname{adj} A\right)}_{\substack{\| \\ A^{-1}}} = I, \qquad \text{and} \qquad \underbrace{\left(\frac{1}{ad - bc} \operatorname{adj} A\right)}_{\substack{\| \\ A^{-1}}} A = I.$$

So, we indeed arrive at

$$\boxed{AA^{-1} = I, \qquad \text{and} \qquad A^{-1}A = I.}$$

To highlight the result:

Fact 2. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, suppose

$$\det A \neq 0, \quad \text{namely,} \quad ad - bc \neq 0.$$

Then

$$\boxed{AA^{-1} = I, \qquad \text{and} \qquad A^{-1}A = I.}$$

- Analogy:

"*We always have*

(**) $$a\,a^{-1} = 1, \qquad \textit{and} \qquad a^{-1}\,a = 1$$

for any number a, provided $a \neq 0$. (Right?) In the same token,

(##) $$A\,A^{-1} = I, \qquad \textit{and} \qquad A^{-1}\,A = I$$

for any matrix A, provided $\det A \neq 0$. These two, (**) and (##), are entirely parallel."

- **The next level question.** Consider two matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}.$$

Suppose $AB = I$. Then is it true

(1) $$BA = I?$$

(2) $$B = A^{-1}, \qquad \text{and} \qquad A = B^{-1}?$$

Answers. 'True', for both part (1) and part (2).

Proof of this requires a careful analysis. We know that the following is true:

(%) "Suppose a and b are both numbers $\big($scalars$\big)$. Suppose $ab = 1$. Then $b = a^{-1}$, and $a = b^{-1}$."

Our question is a generalization of (%). So we extrapolate how we prove (%).

Extrapolation.

"First, $ab = 1$ forces a to be non-zero. Thus a^{-1} exists. Then multiply a^{-1} to the two sides of $ab = 1$: Then you immediately obtain $b = a^{-1}$. Similarly, by multiplying b^{-1} instead of a^{-1} you will obtain $a = b^{-1}$. The very same logic can be employed for matrices to pull the same conclusion for matrices, save that there are a couple of points which prove to be subtle $\big($#1 and #2 below$\big)$."

Point of subtlety # 1: The extrapolation of the part

- " $ab = 1$ *forces* a *to be non-zero*".

The right extrapolation of this statement for matrices is

- " $AB = I$ *forces* A *to have a non-zero determinant.*"

This latter statement is true. However, it is not that trivial. We need to provide a proof of it. $\Big($Here we go again!$\Big)$ For that matter, we in turn need to rely on a so-called "Product Formula".

Point of subtlety # 2: The extrapolation of the part

- "*multiply* a^{-1} to *the two sides of* $ab = 1$ *to get* $b = a^{-1}$."

The right extrapolation of this statement for matrices is

- "*multiply* A^{-1} to *the two sides of* $AB = I$ *to get* $B = A^{-1}$."

A couple of delicate points here: First you need to say you multiply A^{-1} $\underline{\underline{\text{from the left,}}}$ as in

$$A^{-1}\Big(AB\Big) = A^{-1}I.$$

Second, you want to say $A^{-1}\Big(AB\Big)$ is reduced to B. However, technically speaking, in order to be able to safely claim that, you need to know in advance

$$A^{-1}\Big(AB\Big) = \Big(A^{-1}A\Big)B.$$

This turns out to be true, indeed, more generally,

$$A\Big(BC\Big) = \Big(AB\Big)C$$

holds true for three matrices A, B and C. Now, this last cited fact is something that requires a proof. $\Big($Here we go again!$\Big)$ This property $A\Big(BC\Big) = \Big(AB\Big)C$ is called the "Associativity Law".

Formula 1 (Product Formula for 2×2).

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, we have

$$\boxed{\det\Big(AB\Big) = \Big(\det A\Big)\Big(\det B\Big).}$$

Example (that is in sync with Product Formula). Let

$$A = \begin{bmatrix} 2 & 1 \\ -4 & 7 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 6 & -3 \\ 5 & -1 \end{bmatrix}.$$

(These two are just randomly picked.) Calculate $\det A$ and $\det B$:

$$\begin{aligned} \det A &= \begin{vmatrix} 2 & 1 \\ -4 & 7 \end{vmatrix} \\ &= 2 \cdot 7 - 1 \cdot \big(-4\big) = 18, \end{aligned}$$

$$\begin{aligned} \det B &= \begin{vmatrix} 6 & -3 \\ 5 & -1 \end{vmatrix} \\ &= 6 \cdot \big(-1\big) - \big(-3\big) \cdot 5 = 9. \end{aligned}$$

Independently of these,

$$\begin{aligned} AB &= \begin{bmatrix} 2 & 1 \\ -4 & 7 \end{bmatrix} \begin{bmatrix} 6 & -3 \\ 5 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot 6 + 1 \cdot 5 & 2 \cdot \big(-3\big) + 1 \cdot \big(-1\big) \\ \big(-4\big) \cdot 6 + 7 \cdot 5 & \big(-4\big) \cdot \big(-3\big) + 7 \cdot \big(-1\big) \end{bmatrix} \\ &= \begin{bmatrix} 17 & -7 \\ 11 & 5 \end{bmatrix}. \end{aligned}$$

Thus

$$\det\Big(AB\Big) \;=\; 17 \cdot 5 \;-\; \Big(-7\Big) \cdot 11 \;=\; 162.$$

To summarize,

$$\det A \;=\; 18, \qquad \det B \;=\; 9, \qquad \det\Big(A\,B\Big) \;=\; 162.$$

We have $18 \cdot 9 = 162$, and this is consistent with the formula

$$\det\Big(AB\Big) = \Big(\det A\Big)\Big(\det B\Big).$$

Example (that is in sync with Product Formula). Let

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}.$$

(Again these are random choices.)

$$\det A \;=\; \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix}$$

$$= \; 1 \cdot \Big(-1\Big) \;-\; 3 \cdot 2 \;=\; -7,$$

$$\det B \;=\; \begin{vmatrix} 2 & 1 \\ 4 & 5 \end{vmatrix}$$

$$= \; 2 \cdot 5 \;-\; 1 \cdot 4 \;=\; 6.$$

Independently of these,

$$AB \;=\; \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot 2 + 3 \cdot 4 & 1 \cdot 1 + 3 \cdot 5 \\ 2 \cdot 2 + \Big(-1\Big) \cdot 4 & 2 \cdot 1 + \Big(-1\Big) \cdot 5 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 16 \\ 0 & -3 \end{bmatrix}.$$

Thus

$$\det\Big(AB\Big) \;=\; 14\cdot\big(-3\big) \;-\; 16\cdot 0 \;=\; -42.$$

To summarize:

$$\det A \;=\; -7, \qquad \det B \;=\; 6, \qquad \det\Big(A\,B\Big) \;=\; -42.$$

We have $\Big(-7\Big)\cdot 6 = -42,$ and this is consistent with the formula:

$$\det\Big(AB\Big) = \Big(\det A\Big)\Big(\det B\Big).$$

- We have to prove the statement using a general pair of matrices A and B :

$$A \;=\; \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{and} \qquad B \;=\; \begin{bmatrix} p & q \\ r & s \end{bmatrix},$$

where a, b, c, d, p, q, r and s are arbitrary.

- On the one hand

$$\det A \;=\; \begin{vmatrix} a & b \\ c & d \end{vmatrix} \;=\; ad \,-\, bc, \qquad \det B \;=\; \begin{vmatrix} p & q \\ r & s \end{vmatrix} \;=\; ps \,-\, qr,$$

and on the other hand

$$A\,B \;=\; \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix},$$

so

$$\begin{aligned} \det\Big(A\,B\Big) \;&=\; \begin{vmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{vmatrix} \\ &=\; \Big(ap + br\Big)\Big(cq + ds\Big) \,-\, \Big(aq + bs\Big)\Big(cp + dr\Big), \end{aligned}$$

so the statement is equivalent to the following:

Formula 1′ (**Product Formula,** **spelt-out version,** **2 × 2**).

$$(*) \qquad \boxed{\begin{aligned} &\Big(ap + br\Big)\Big(cq + ds\Big) - \Big(aq + bs\Big)\Big(cp + dr\Big) \\ &\qquad\qquad = \Big(ad - bc\Big)\Big(ps - qr\Big). \end{aligned}}$$

Agree with the following:

" *In order to prove* Formula 1, *it suffices to prove* $(*)$ (Formula 1′). "

Proof of (∗) (Formula 1′).

The left-hand side of $(*)$

$$\begin{aligned}
&= \Big(ap + br\Big)\Big(cq + ds\Big) - \Big(aq + bs\Big)\Big(cp + dr\Big) \\
&= \Big(apcq + apds + brcq + brds\Big) - \Big(aqcp + aqdr + bscp + bsdr\Big) \\
&= \Big(acpq + adps + bcqr + bdrs\Big) - \Big(acpq + adqr + bcps + bdrs\Big) \\
&= adps + bcqr - adqr - bcps.
\end{aligned}$$

The right-hand side of $(*)$

$$\begin{aligned}
&= \Big(ad - bc\Big)\Big(ps - qr\Big) \\
&= adps - adqr - bcps + bcqr \\
&= adps + bcqr - adqr - bcps.
\end{aligned}$$

The above calculations show that the two sides of $(*)$ are equal. □

Exercise (= "VI"; Exercise 1). $A = \begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix}$, and $B = \begin{bmatrix} 6 & 5 \\ 8 & 3 \end{bmatrix}$, calculate

(1) $\det A$, (2) $\det B$, (3) $\left(\det A\right)\left(\det B\right)$ based on (1–2),

(4) AB, and (5) $\det\left(AB\right)$ based on (4).

Confirm that the answer for (3) and the answer for (5) coincide.

• We have just proved Product Formula. You may prematurely conclude that the subject is "*ad nauseum*". The truth is, the above formula has larger size counterparts, and those will not be as rudimentary. Indeed, take a quick peek at how each of the 3×3 and the 4×4 counterparts looks like $\big($below$\big)$. They probably don't strike you as trivial.

Product Formula (Spelt-out version, $\mathbf{3 \times 3}$).

$$
\begin{aligned}
&\Big(a_1p_1 + a_2p_2 + a_3p_3 \Big)\Big(b_1q_1 + b_2q_2 + b_3q_3 \Big)\Big(c_1r_1 + c_2r_2 + c_3r_3 \Big) \\
&+ \Big(a_1q_1 + a_2q_2 + a_3q_3 \Big)\Big(b_1r_1 + b_2r_2 + b_3r_3 \Big)\Big(c_1p_1 + c_2p_2 + c_3p_3 \Big) \\
&+ \Big(a_1r_1 + a_2r_2 + a_3r_3 \Big)\Big(b_1p_1 + b_2p_2 + b_3p_3 \Big)\Big(c_1q_1 + c_2q_2 + c_3q_3 \Big) \\
&- \Big(a_1r_1 + a_2r_2 + a_3r_3 \Big)\Big(b_1q_1 + b_2q_2 + b_3q_3 \Big)\Big(c_1p_1 + c_2p_2 + c_3p_3 \Big) \\
&- \Big(a_1p_1 + a_2p_2 + a_3p_3 \Big)\Big(b_1r_1 + b_2r_2 + b_3r_3 \Big)\Big(c_1q_1 + c_2q_2 + c_3q_3 \Big) \\
&- \Big(a_1q_1 + a_2q_2 + a_3q_3 \Big)\Big(b_1p_1 + b_2p_2 + b_3p_3 \Big)\Big(c_1r_1 + c_2r_2 + c_3r_3 \Big) \\
&\qquad = \Big(a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3 \Big) \\
&\qquad\quad \cdot \Big(p_1q_2r_3 + p_2q_3r_1 + p_3q_1r_2 - p_3q_2r_1 - p_1q_3r_2 - p_2q_1r_3 \Big) .
\end{aligned}
$$

Product Formula (Spelt-out version, 4×4).

$$
\begin{aligned}
&(a_1p_1+a_2q_1+a_3r_1+a_4s_1)(b_1p_2+b_2q_2+b_3r_2+b_4s_2)(c_1p_3+c_2q_3+c_3r_3+c_4s_3)(d_1p_4+d_2q_4+d_3r_4+d_4s_4)\\
&+(a_1p_1+a_2q_1+a_3r_1+a_4s_1)(b_1p_3+b_2q_3+b_3r_3+b_4s_3)(c_1p_4+c_2q_4+c_3r_4+c_4s_4)(d_1p_2+d_2q_2+d_3r_2+d_4s_2)\\
&+(a_1p_1+a_2q_1+a_3r_1+a_4s_1)(b_1p_4+b_2q_4+b_3r_4+b_4s_4)(c_1p_2+c_2q_2+c_3r_2+c_4s_2)(d_1p_3+d_2q_3+d_3r_3+d_4s_3)\\
&+(a_1p_2+a_2q_2+a_3r_2+a_4s_2)(b_1p_1+b_2q_1+b_3r_1+b_4s_1)(c_1p_4+c_2q_4+c_3r_4+c_4s_4)(d_1p_3+d_2q_3+d_3r_3+d_4s_3)\\
&+(a_1p_2+a_2q_2+a_3r_2+a_4s_2)(b_1p_4+b_2q_4+b_3r_4+b_4s_4)(c_1p_3+c_2q_3+c_3r_3+c_4s_3)(d_1p_1+d_2q_1+d_3r_1+d_4s_1)\\
&+(a_1p_2+a_2q_2+a_3r_2+a_4s_2)(b_1p_3+b_2q_3+b_3r_3+b_4s_3)(c_1p_1+c_2q_1+c_3r_1+c_4s_1)(d_1p_4+d_2q_4+d_3r_4+d_4s_4)\\
&+(a_1p_3+a_2q_3+a_3r_3+a_4s_3)(b_1p_1+b_2q_1+b_3r_1+b_4s_1)(c_1p_2+c_2q_2+c_3r_2+c_4s_2)(d_1p_4+d_2q_4+d_3r_4+d_4s_4)\\
&+(a_1p_3+a_2q_3+a_3r_3+a_4s_3)(b_1p_2+b_2q_2+b_3r_2+b_4s_2)(c_1p_4+c_2q_4+c_3r_4+c_4s_4)(d_1p_1+d_2q_1+d_3r_1+d_4s_1)\\
&+(a_1p_3+a_2q_3+a_3r_3+a_4s_3)(b_1p_4+b_2q_4+b_3r_4+b_4s_4)(c_1p_1+c_2q_1+c_3r_1+c_4s_1)(d_1p_2+d_2q_2+d_3r_2+d_4s_2)\\
&+(a_1p_4+a_2q_4+a_3r_4+a_4s_4)(b_1p_1+b_2q_1+b_3r_1+b_4s_1)(c_1p_3+c_2q_3+c_3r_3+c_4s_3)(d_1p_2+d_2q_2+d_3r_2+d_4s_2)\\
&+(a_1p_4+a_2q_4+a_3r_4+a_4s_4)(b_1p_3+b_2q_3+b_3r_3+b_4s_3)(c_1p_2+c_2q_2+c_3r_2+c_4s_2)(d_1p_1+d_2q_1+d_3r_1+d_4s_1)\\
&+(a_1p_4+a_2q_4+a_3r_4+a_4s_4)(b_1p_2+b_2q_2+b_3r_2+b_4s_2)(c_1p_1+c_2q_1+c_3r_1+c_4s_1)(d_1p_3+d_2q_3+d_3r_3+d_4s_3)\\
&-(a_1p_1+a_2q_1+a_3r_1+a_4s_1)(b_1p_2+b_2q_2+b_3r_2+b_4s_2)(c_1p_4+c_2q_4+c_3r_4+c_4s_4)(d_1p_3+d_2q_3+d_3r_3+d_4s_3)\\
&-(a_1p_1+a_2q_1+a_3r_1+a_4s_1)(b_1p_4+b_2q_4+b_3r_4+b_4s_4)(c_1p_3+c_2q_3+c_3r_3+c_4s_3)(d_1p_2+d_2q_2+d_3r_2+d_4s_2)\\
&-(a_1p_1+a_2q_1+a_3r_1+a_4s_1)(b_1p_3+b_2q_3+b_3r_3+b_4s_3)(c_1p_2+c_2q_2+c_3r_2+c_4s_2)(d_1p_4+d_2q_4+d_3r_4+d_4s_4)\\
&-(a_1p_2+a_2q_2+a_3r_2+a_4s_2)(b_1p_1+b_2q_1+b_3r_1+b_4s_1)(c_1p_3+c_2q_3+c_3r_3+c_4s_3)(d_1p_4+d_2q_4+d_3r_4+d_4s_4)\\
&-(a_1p_2+a_2q_2+a_3r_2+a_4s_2)(b_1p_3+b_2q_3+b_3r_3+b_4s_3)(c_1p_4+c_2q_4+c_3r_4+c_4s_4)(d_1p_1+d_2q_1+d_3r_1+d_4s_1)\\
&-(a_1p_2+a_2q_2+a_3r_2+a_4s_2)(b_1p_4+b_2q_4+b_3r_4+b_4s_4)(c_1p_1+c_2q_1+c_3r_1+c_4s_1)(d_1p_3+d_2q_3+d_3r_3+d_4s_3)\\
&-(a_1p_3+a_2q_3+a_3r_3+a_4s_3)(b_1p_1+b_2q_1+b_3r_1+b_4s_1)(c_1p_4+c_2q_4+c_3r_4+c_4s_4)(d_1p_2+d_2q_2+d_3r_2+d_4s_2)\\
&-(a_1p_3+a_2q_3+a_3r_3+a_4s_3)(b_1p_4+b_2q_4+b_3r_4+b_4s_4)(c_1p_2+c_2q_2+c_3r_2+c_4s_2)(d_1p_1+d_2q_1+d_3r_1+d_4s_1)\\
&-(a_1p_3+a_2q_3+a_3r_3+a_4s_3)(b_1p_2+b_2q_2+b_3r_2+b_4s_2)(c_1p_1+c_2q_1+c_3r_1+c_4s_1)(d_1p_4+d_2q_4+d_3r_4+d_4s_4)\\
&-(a_1p_4+a_2q_4+a_3r_4+a_4s_4)(b_1p_1+b_2q_1+b_3r_1+b_4s_1)(c_1p_2+c_2q_2+c_3r_2+c_4s_2)(d_1p_3+d_2q_3+d_3r_3+d_4s_3)\\
&-(a_1p_4+a_2q_4+a_3r_4+a_4s_4)(b_1p_2+b_2q_2+b_3r_2+b_4s_2)(c_1p_3+c_2q_3+c_3r_3+c_4s_3)(d_1p_1+d_2q_1+d_3r_1+d_4s_1)\\
&-(a_1p_4+a_2q_4+a_3r_4+a_4s_4)(b_1p_3+b_2q_3+b_3r_3+b_4s_3)(c_1p_1+c_2q_1+c_3r_1+c_4s_1)(d_1p_2+d_2q_2+d_3r_2+d_4s_2)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \Big(\; a_1b_2c_3d_4+a_1b_3c_4d_2+a_1b_4c_2d_3+a_2b_1c_4d_3+a_2b_4c_3d_1+a_2b_3c_1d_4\\
&+a_3b_1c_2d_4+a_3b_2c_4d_1+a_3b_4c_1d_2+a_4b_1c_3d_2+a_4b_3c_2d_1+a_4b_2c_1d_3\\
&-a_1b_2c_4d_3-a_1b_4c_3d_2-a_1b_3c_2d_4-a_2b_1c_3d_4-a_2b_3c_4d_1-a_2b_4c_1d_3\\
&-a_3b_1c_4d_2-a_3b_4c_2d_1-a_3b_2c_1d_4-a_4b_1c_2d_3-a_4b_2c_3d_1-a_4b_3c_1d_2 \; \Big)\\
\cdot \quad & \Big(\; p_1q_2r_3s_4+p_1q_3r_4s_2+p_1q_4r_2s_3+p_2q_1r_4s_3+p_2q_4r_3s_1+p_2q_3r_1s_4\\
&+p_3q_1r_2s_4+p_3q_2r_4s_1+p_3q_4r_1s_2+p_4q_1r_3s_2+p_4q_3r_2s_1+p_4q_2r_1s_3\\
&-p_1q_2r_4s_3-p_1q_4r_3s_2-p_1q_3r_2s_4-p_2q_1r_3s_4-p_2q_3r_4s_1-p_2q_4r_1s_3\\
&-p_3q_1r_4s_2-p_3q_4r_2s_1y-p_3q_2r_1s_4-p_4q_1r_2s_3-p_4q_2r_3s_1-p_4q_3r_1s_2 \; \Big).
\end{aligned}
$$

Now, suppose I asked you to *prove* these, you probably cannot think of any immediate way to go about. If you say your computer can handle it, I say this: The above is only up to 4×4. The same formula for 5×5, 6×6, 7×7, $\cdots$ exist. The expansion of the left-hand side, and the right-hand side, of the $n \times n$ counterpart formula, before cancellations, involve the following number of terms:

$$n^n \cdot n!, \qquad \text{and} \qquad \left(n!\right)^2,$$

respectively. These numbers for $n = 3, 4, 5, 6, 7, 8$ come out as

$$\begin{array}{lcrr}
n = 3 & \Longrightarrow & 162, & 36. \\
n = 4 & \Longrightarrow & 6144, & 576. \\
n = 5 & \Longrightarrow & 375000, & 14400. \\
n = 6 & \Longrightarrow & 33592320, & 518400. \\
n = 7 & \Longrightarrow & 4150656720, & 25401600. \\
n = 8 & \Longrightarrow & 676457349120, & 1625702400. \\
\vdots & & \vdots & \vdots
\end{array}$$

These are the numbers of terms your computer is supposed to deal with. These numbers grow exponentially as n grows. As you can read off from the above table, already for $n = 8$ the number $n^n \cdot n!$ is a 12-digit number (an order of trillion). For $n = 100$ the corresponding number $n^n \cdot n!$ is a 358-digit number. Sooner or later it will go above your computer's capability. Now, be that as it may, any mathematical software you've heard of actually knows the Product Formula in any size n. The next thing I say is important: That's because the formula is known to be true, by humans, because human mathematicians have logically proved it, in an *ex machina* way. Then whoever came up with that software (or whoever is in charge of updating that software) borrowed that knowledge and installed the formula on the software. The computer itself does not have enough intelligence to generate that proof of the formula. How to prove the formula, in an *ex machina* way, and things of this nature, are what you are going to learn in this class: You are going to work on things computers cannot replace. So far we haven't even defined the determinants for matrices larger than 3×3, or matrix multiplication for matrices larger than 2×2. We need to do that first. Then how to prove Product Formula for $n \times n$ is a whole different story altogether. That's coming up. For the rest of today we switch gears.