Math 290 ELEMENTARY LINEAR ALGEBRA

STUDY GUIDE FOR MIDTERM EXAM – B

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§6. Product Formula. Associativity Law (Continued).

• **Associativity Law.**

For three matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}, \quad C = \begin{bmatrix} x & y \\ z & w \end{bmatrix},$$

it doesn't matter whether you

(i) calculate ABC as $\Big(AB\Big)C$, or

(ii) calculate ABC as $A\Big(BC\Big)$,

you will end up getting the same answer.

Formula 2 (Associativity Law). For

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}, \qquad C = \begin{bmatrix} x & y \\ z & w \end{bmatrix},$$

we have

$$\boxed{\Big(AB\Big)C = A\Big(BC\Big).}$$

Proof.

$$\Big(AB\Big)C \;=\; \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix}\right) \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

$$= \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

$$= \begin{bmatrix} apx + brx + aqz + bsz & apy + bry + aqw + bsw \\ cpx + drx + cqz + dsz & cpy + dry + cqw + dsw \end{bmatrix},$$

and

$$A\Big(BC\Big) \;=\; \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix}\right)$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} px + qz & py + qw \\ rx + sz & ry + sw \end{bmatrix}$$

$$= \begin{bmatrix} apx + aqz + brx + bsz & apy + aqw + bry + bsw \\ cpx + cqz + drx + dsz & cpy + cqw + dry + dsw \end{bmatrix}.$$

These two outcomes indeed coincide. $\square$

- We know that the same is true for numbers, as in

$$\Big(2 \cdot 3\Big) \cdot 5 \;=\; 2 \cdot \Big(3 \cdot 5\Big).$$

More generally, if a, b and c are numbers $\big($real numbers, to be precise$\big)$, then

$$\Big(a \cdot b\Big) \cdot c \;=\; a \cdot \Big(b \cdot c\Big).$$

Granted, it is still imperative that you pay heed to the above formula, and learn how to prove it, for at least two different reasons.

Reason 1. When you generalize something from numbers to matrices, you are going to lose some of the properties. In general $AB \neq BA$ for matrices A and B. Needless to say, $ab = ba$ for numbers a and b. So you need to keep track of both (a) those properties that are carried over from numbers to matrices, and (b) those that aren't.

Reason 2. There is actually a number system wherein

$$\Big(a \cdot b\Big) \cdot c \neq a \cdot \Big(b \cdot c\Big).$$

(octonion numbers / octonions / Cayley numbers). You don't have to know what that is. On the other hand, quaternions are a must. They are more basic than octonions. We are going to address quaternions in due course.

Definition. Define $\boxed{ABC}$ as either $A\Big(BC\Big)$, or equivalently, $\Big(AB\Big)C$.

- **Multiplications of four or more matrices.** For

$$A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \quad B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}, \quad C = \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix}, \quad D = \begin{bmatrix} a_4 & b_4 \\ c_4 & d_4 \end{bmatrix},$$

it doesn't matter whether you

(i) calculate $ABCD$ as $\Big(\big(AB\big)C\Big)D$,

(ii) calculate $ABCD$ as $\Big(A\big(BC\big)\Big)D$,

(iii) calculate $ABCD$ as $A\Big(\big(BC\big)D\Big)$,

(iv) calculate $ABCD$ as $A\Big(B\big(CD\big)\Big)$, or

(v) calculate $ABCD$ as $\Big(AB\Big)\Big(CD\Big)$.

You will end up getting the same answer.

Exercise (= "VI"; Exercise 2). Explain why (i–v) all coincide.

$\Big[\underline{\textbf{Hint}}\Big]$ **:** First explain why (i) and (ii) are the same. For that matter, it suffices to say (i) and (ii) are both $\big(ABC\big)D$. Next, explain why (iii) and (iv) are the same $\big($same logic$\big)$. Next, explain why (i) and (v) are the same $\big($set $AB = E\big)$. Finally, explain why (iv) and (v) are the same $\big($set $CD = F\big)$.

Corollary. Let $A,\ B,\ C,\ D$ be as above. Then

$$\Big(\big(AB\big)C\Big)D \;=\; \Big(A\big(BC\big)\Big)D \;=\; A\Big(\big(BC\big)D\Big)$$
$$=\; A\Big(B\big(CD\big)\Big) \;=\; \Big(AB\Big)\Big(CD\Big).$$

Definition. Define $\boxed{ABCD}$ as

$$ABCD \;=\; \Big(\big(AB\big)C\Big)D \;=\; \Big(A\big(BC\big)\Big)D \;=\; A\Big(\big(BC\big)D\Big)$$
$$=\; A\Big(B\big(CD\big)\Big) \;=\; \Big(AB\Big)\Big(CD\Big).$$

- **Consecutive product.** Similarly, for

$$A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix}, \quad \cdots, \quad A_k = \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix},$$

define $\boxed{A_1\,A_2\,A_3\,\cdots A_{k-1}A_k}$ as

$$A_1\,A_2\,A_3\,\cdots A_{k-1}A_k \;=\; \bigg(\Big(\big(\,\cdot\cdot\big(\big(A_1\,A_2\big)\,A_3\big)\,\cdots\,\big)A_{k-2}\Big)\,A_{k-1}\bigg)\,A_k$$
$$=\; A_1\bigg(A_2\,\Big(A_3\,\big(\,\cdots\,\big(A_{k-2}\,\big(A_{k-1}\,A_k\big)\big)\,\cdot\cdot\big)\Big)\bigg).$$

Exercise (= "VI"; Exercise 3). Let

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Calculate

(1) AB. (2) BC. (3) CD.
(4) ABC. (5) BCD. (6) $ABCD$.

- **Powers.** As a special case of

$$A_1 A_2 A_3 \ \cdots \ A_k,$$

we may consider the following: For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, define

$$A^1 = A,$$

$$A^2 = AA,$$

$$A^3 = AAA,$$

$$A^4 = AAAA,$$

$$\vdots$$

$$A^k = \underbrace{A\ A\ A\ \cdots\ A}_{k}.$$

Example3. For $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$$I^k = I, \quad \text{for} \quad k = 1, 2, 3, \cdots .$$

Paraphrase:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{for} \quad k = 1, 2, 3, \cdots .$$

- More formulas.

Formula 3. Let t and u be scalars. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}.$$

Then

$$\Big(tA\Big)\Big(uB\Big) = (tu)\Big(AB\Big).$$

Corollary. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and t a scalar. Then for $k = 1, 2, 3, \cdots,$

$$\Big(t\,A\Big)^k = t^k\,A^k.$$

Example. In Corollary above, if you set $A = I$, then

$$A = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} \qquad \Longrightarrow \qquad A^k = \begin{bmatrix} t^k & 0 \\ 0 & t^k \end{bmatrix}.$$

Exercise (= "VI"; Exercise 4). Let $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. For $k = 1, 2, 3, \cdots,$ prove

$(*)_k$
$$A^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix},$$

via mathematical induction. Practically, do (i) and (ii) below:

(i) Prove that $(*)_1$ $\big(= (*)_k$ for $k = 1\big)$ is true.

(ii) Assume that $(*)_k$ is true, and with that assumption prove that $(*)_{k+1}$ is true.

Exercise (= "VI"; Exercise 5). For $k = 1, 2, 3, \cdots,$ find

(1) $\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}^k$. (2) $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^k$. (3) $\begin{bmatrix} 2 & a \\ 0 & 2 \end{bmatrix}^k$.

[**Hint for (2)**] : First verify $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix}$.

Exercise (= "VI"; Exercise 6).

(1) True or False. $\left(AB\right)^2 = A^2B^2$.

(2) Suppose $AB = BA$. True or False. $\left(AB\right)^2 = A^2B^2$.

§7. Matrix addition & subtraction. Distributive Law.

Definition (Matrix addition/subtraction).

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, define

$$\boxed{A + B = \begin{bmatrix} a+p & b+q \\ c+r & d+s \end{bmatrix},}$$

$$\boxed{A - B = \begin{bmatrix} a-p & b-q \\ c-r & d-s \end{bmatrix}.}$$

Example. For $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & -2 \\ 4 & 2 \end{bmatrix}$, we have

$$A + B = \begin{bmatrix} 1 + \left(-3\right) & 2 + \left(-2\right) \\ 2 + 4 & 1 + 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 6 & 3 \end{bmatrix},$$

$$A - B = \begin{bmatrix} 1 - \left(-3\right) & 2 - \left(-2\right) \\ 2 - 4 & 1 - 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ -2 & -1 \end{bmatrix}.$$

Warning 1. In general,

$$\boxed{\det A + \det B \quad \underline{\underline{\text{and}}} \quad \det\left(A+B\right) \quad \underline{\underline{\text{are not equal.}}}}$$

$$\boxed{\det A - \det B \quad \underline{\underline{\text{and}}} \quad \det\left(A-B\right) \quad \underline{\underline{\text{are not equal.}}}}$$

Exercise (= "VII"; Exercise 1). For $A = \begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix}$, and $B = \begin{bmatrix} 2 & 6 \\ 2 & 3 \end{bmatrix}$, calculate

(1) $A + B$, (2) $\det\left(A+B\right)$ based on (1), (3) $\det A + \det B$.

Do the answer for (2) and the answer for (3) coincide? Also calculate

(4) $A - B$, (5) $\det\left(A-B\right)$ based on (4), (6) $\det A - \det B$.

Do the answer for (5) and the answer for (6) coincide?

Quiz. Recall $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Let $A = \begin{bmatrix} 3 & 1 \\ 4 & 6 \end{bmatrix}$. Write out $\lambda I - A$.

Solution. Since

$$\lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \qquad \text{and} \qquad A = \begin{bmatrix} 3 & 1 \\ 4 & 6 \end{bmatrix},$$

So

$$\lambda I - A = \begin{bmatrix} \lambda-3 & 0-1 \\ 0-4 & \lambda-6 \end{bmatrix} = \begin{bmatrix} \lambda-3 & -1 \\ -4 & \lambda-6 \end{bmatrix}.$$

Quiz. (1) Calculate the determinant of $\lambda I - A = \begin{bmatrix} \lambda-3 & -1 \\ -4 & \lambda-6 \end{bmatrix}$.

(2) Solve the equation $\begin{vmatrix} \lambda-3 & -1 \\ -4 & \lambda-6 \end{vmatrix} = 0.$

Solution. (1): $\begin{vmatrix} \lambda-3 & -1 \\ -4 & \lambda-6 \end{vmatrix} = \left(\lambda - 3\right)\left(\lambda - 6\right) - \left(-1\right) \cdot \left(-4\right)$

$$= \left(\lambda^2 - 9\lambda + 18\right) - 4 = \lambda^2 - 9\lambda + 14.$$

(2) Factor $\lambda^2 - 9\lambda + 14$:

$$\lambda^2 - 9\lambda + 14 = \Big(\lambda - 2\Big)\Big(\lambda - 7\Big).$$

So the roots for the equation

$$\begin{vmatrix} \lambda - 3 & -1 \\ -4 & \lambda - 6 \end{vmatrix} = 0$$

are

$$\lambda = 2, \qquad \text{and} \qquad \lambda = 7. \quad \square$$

- $\lambda = 2$, $\lambda = 7$ are called the eigenvalues of the matrix $A = \begin{bmatrix} 3 & 1 \\ 4 & 6 \end{bmatrix}$.
- $\det\Big(\lambda I - A\Big) = \lambda^2 - 9\lambda + 14$ is called the characteristic polynomial of A.
- $\lambda^2 - 9\lambda + 14 = 0$ is called the characteristic equation of A.

Terminology. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. (i) the **characteristic polynomial** of A means $\boxed{\det\Big(\lambda I - A\Big),}$ or the same to say, $\boxed{\begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix}.}$

(ii) the characteristic **equation** of A means $\boxed{\det\Big(\lambda I - A\Big) = 0,}$ or the same to say, $\boxed{\begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = 0.}$

(iii) the **eigenvalues** of A mean $\boxed{\text{the } \textbf{roots} \text{ of } \det\Big(\lambda I - A\Big) = 0,}$ or the same to say, $\boxed{\text{the } \textbf{roots} \text{ of } \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = 0.}$

Warning 2. Don't ever simplify $\det\left(\lambda I - A\right)$ as $\det\left(\lambda I\right) - \det A$. That would be incorrect. Also, a formation like $\det\left(\lambda I\right) \;-\; \det A$ is of little significance.

Notation $\boldsymbol{\chi_A\left(\lambda\right)}$ for the characteristic polynomial of $\boldsymbol{A}$.

From now on, we denote the characteristic polynomial of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as

$$\boxed{\chi_A\left(\lambda\right).}$$

So,

$$\boxed{\chi_A\left(\lambda\right) \;=\; \det\left(\lambda I - A\right).}$$

Or, the same to say

$$\boxed{\chi_A\left(\lambda\right) \;=\; \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix}\,.}$$

Example. Find $\chi_A\left(\lambda\right)$ for $A = \begin{bmatrix} 8 & 3 \\ 6 & 5 \end{bmatrix}$. Find the eigenvalues of A.

— Well, this is a piece of cake. Here we go:

$$\begin{aligned} \chi_A\left(\lambda\right) \;&=\; \begin{vmatrix} \lambda - 8 & -3 \\ -6 & \lambda - 5 \end{vmatrix} \\ &=\; \left(\lambda - 8\right)\left(\lambda - 5\right) \;-\; \left(-3\right)\cdot\left(-6\right) \\ &=\; \lambda^2 \;-\; 13\lambda \;+\; 22. \end{aligned}$$

This is factored as

$$\chi_A\left(\lambda\right) \;=\; \left(\lambda - 2\right)\left(\lambda - 11\right),$$

so accordingly the eigenvalues of A are found as $\lambda = 2$, and $\lambda = 11$.

Exercise (= "VII"; Exercise 2). Find $\chi_A(\lambda)$. Find the eigenvalues of A.

(1) $A = \begin{bmatrix} 6 & 4 \\ 6 & 1 \end{bmatrix}$. (2) $A = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{3}{2} & \frac{5}{2} \end{bmatrix}$.

Formula 1 (Distributive Laws). For

$$A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \quad B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}, \quad C = \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix}, \quad D = \begin{bmatrix} a_4 & b_4 \\ c_4 & d_4 \end{bmatrix},$$

the following hold:

(1) $$A\Big(B+C\Big) = AB + AC,$$

(2) $$\Big(B+C\Big)D = BD + CD,$$

(3) $$A\Big(B+C\Big)D = ABD + ACD,$$

(4) $$A\Big(B-C\Big) = AB - AC,$$

(5) $$\Big(B-C\Big)D = BD - CD,$$

(6) $$A\Big(B-C\Big)D = ABD - ACD.$$

Corollary 1. Let A, B, C and D be as above. Then

$$\boxed{\Big(A+B\Big)\Big(C+D\Big) = AC + AD + BC + BD.}$$

Exercise (= "VII"; Exercise 3).

(a) Prove parts (1–2) of Formula 1 above. As for part (1), calculate each of

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} + \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix} \right), \qquad \text{and}$$

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix}$$

separately, and verify that they match. Part (2) is completely similar.

(b) Prove that part (3) of Formula 1 <u>is equivalent to</u> parts (1–2) of the same formula. So, prove "(1) implies (2–3)" <u>and</u> "(2–3) imply (1)" both.

Corollary 2. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$,

$$\boxed{\Big(A+B\Big)^2 = A^2 + AB + BA + B^2.}$$

Formula 2 (Distributive & Associative Laws Involving Scalars – I). For

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix},$$

the following hold:

(1) $$-\Big(A+B\Big) = -A - B.$$

(2) $$\Big(-A\Big)B = -\Big(AB\Big).$$

(3) $$A\Big(-B\Big) = -\Big(AB\Big).$$

More generally, for a scalar t,

(4) $$t\Big(A+B\Big) = tA + tB.$$

(5) $$\Big(tA\Big)B = t\Big(AB\Big).$$

(6) $$A\Big(tB\Big) = t\Big(AB\Big).$$

Corollary 3. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$,

$$\boxed{\Big(A+B\Big)\Big(A-B\Big) = A^2 + BA - AB - B^2.}$$

Example. $$A\Big(I-B\Big)C \;=\; AIC \;-\; ABC$$
$$=\; AC \;-\; ABC.$$

Example. A tweak:

$$A\Big(2I-B\Big)C \;=\; A\Big(2I\Big)C \;-\; ABC$$
$$=\; 2AC \;-\; ABC.$$

Example. In the above, $C = A^{-1}$:

$$A\Big(2I-B\Big)A^{-1} \;=\; 2AA^{-1} \;-\; ABA^{-1}$$
$$=\; 2I \;-\; ABA^{-1}.$$

Example. $$\Big(ABA^{-1}\Big)^2 \;=\; ABA^{-1}ABA^{-1}$$
$$=\; ABIBA^{-1}$$
$$=\; ABBA^{-1}$$
$$=\; AB^2A^{-1}.$$

Example. Add $4I$:

$$4I + \Big(ABA^{-1}\Big)^2 \;=\; 4I + AB^2A^{-1}.$$

Example. Meanwhile

$$A\Big(4I + B^2\Big)A^{-1} \;=\; A\Big(4I\Big)A^{-1} + AB^2A^{-1}$$
$$=\; 4AIA^{-1} + AB^2A^{-1}$$
$$=\; 4I + AB^2A^{-1}.$$

Example. Combine the last two of the above:

$$(\#) \qquad 4I + \left(ABA^{-1}\right)^2 = A\left(4I + B^2\right)A^{-1}.$$

If you set $f(x) = 4 + x^2$, $(\#)$ is

$$(\#)' \qquad f\left(ABA^{-1}\right) = Af(B)A^{-1}.$$

More on this later.

Formula 3 (Distributive & Associative Laws Involving Scalars – II). For

$$A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \quad B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}, \quad C = \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix}, \quad D = \begin{bmatrix} a_4 & b_4 \\ c_4 & d_4 \end{bmatrix},$$

the following hold:

$$(1) \qquad \left(-A\right)\left(B+C\right) = -AB - AC.$$

$$(2) \qquad \left(B+C\right)\left(-D\right) = -BD - CD.$$

More generally, for a scalar t,

$$(3) \qquad \left(tA\right)\left(B+C\right) = tAB + tAC.$$

$$(4) \qquad \left(B+C\right)\left(tD\right) = tBD + tCD.$$

Also, for scalars t and u,

$$(5) \qquad A\left(tB+uC\right) = tAB + uAC.$$

$$(6) \qquad \left(tB+uC\right)D = tBD + uCD.$$

§8. Matrix multiplication for the 3×3 case.

- **Rule.**

$$\boxed{\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} a_1 p + a_2 q + a_3 r \\ b_1 p + b_2 q + b_3 r \\ c_1 p + c_2 q + c_3 r \end{bmatrix}.}$$

Paraphrase:

$$\boxed{\begin{array}{l} A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \\ \Longrightarrow \qquad A\boldsymbol{x} = \begin{bmatrix} a_1 p + a_2 q + a_3 r \\ b_1 p + b_2 q + b_3 r \\ c_1 p + c_2 q + c_3 r \end{bmatrix}. \end{array}}$$

Break-down.

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \boxed{\diamondsuit} \\ \boxed{\spadesuit} \\ \boxed{\triangle} \end{bmatrix}.$$

(i) To find $\diamondsuit$, observe

$$\begin{bmatrix} \boxed{a_1 \quad a_2 \quad a_3} \\ b_1 \quad b_2 \quad b_3 \\ c_1 \quad c_2 \quad c_3 \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} p \\ q \\ r \end{matrix}} \end{bmatrix} = \begin{bmatrix} \boxed{a_1 p + a_2 q + a_3 r} \\ \boxed{\spadesuit} \\ \boxed{} \end{bmatrix}.$$

(ii) To find ♠, observe

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ \boxed{b_1 \quad b_2 \quad b_3} \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} p \\ q \\ r \end{matrix}} \end{bmatrix} = \begin{bmatrix} \boxed{a_1 p + a_2 q + a_3 r} \\ \boxed{b_1 p + b_2 q + b_3 r} \\ \boxed{\triangle} \end{bmatrix}.$$

(iii) To find $\triangle$, observe

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \boxed{c_1 \quad c_2 \quad c_3} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} p \\ q \\ r \end{matrix}} \end{bmatrix} = \begin{bmatrix} \boxed{a_1 p + a_2 q + a_3 r} \\ \boxed{b_1 p + b_2 q + b_3 r} \\ \boxed{c_1 p + c_2 q + c_3 r} \end{bmatrix}.$$

Example. For $A = \begin{bmatrix} 3 & -6 & 5 \\ -2 & 4 & 7 \\ -1 & 3 & 9 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, we have

$$A\boldsymbol{x} = \begin{bmatrix} 3 & -6 & 5 \\ -2 & 4 & 7 \\ -1 & 3 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \cdot 2 + (-6) \cdot 3 + 5 \cdot 1 \\ (-2) \cdot 2 + 4 \cdot 3 + 7 \cdot 1 \\ (-1) \cdot 2 + 3 \cdot 3 + 9 \cdot 1 \end{bmatrix} = \begin{bmatrix} -7 \\ 15 \\ 16 \end{bmatrix}.$$

Exercise (= "VIII"; Exercise 1). Perform

(1) $\begin{bmatrix} 4 & 0 & 3 \\ 0 & 6 & 5 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}$. (2) $A\boldsymbol{x}$, where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$.

(3) $A\boldsymbol{x}$, where $A = \begin{bmatrix} 7 & 4 & -4 \\ -5 & -2 & 5 \\ 2 & 2 & 3 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix}$.

- Next, two of the 3×3 matrices multiplied together.

Rule. $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix}$ is calculated as

$$\boxed{\begin{bmatrix} a_1p_1 + a_2q_1 + a_3r_1 & a_1p_2 + a_2q_2 + a_3r_2 & a_1p_3 + a_2q_3 + a_3r_3 \\ b_1p_1 + b_2q_1 + b_3r_1 & b_1p_2 + b_2q_2 + b_3r_2 & b_1p_3 + b_2q_3 + b_3r_3 \\ c_1p_1 + c_2q_1 + c_3r_1 & c_1p_2 + c_2q_2 + c_3r_2 & c_1p_3 + c_2q_3 + c_3r_3 \end{bmatrix}.}$$

- Paraphrase:

$$\boxed{\begin{aligned} A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}, \quad B = \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix} \\ \Longrightarrow AB = \begin{bmatrix} a_1p_1 + a_2q_1 + a_3r_1 & a_1p_2 + a_2q_2 + a_3r_2 & a_1p_3 + a_2q_3 + a_3r_3 \\ b_1p_1 + b_2q_1 + b_3r_1 & b_1p_2 + b_2q_2 + b_3r_2 & b_1p_3 + b_2q_3 + b_3r_3 \\ c_1p_1 + c_2q_1 + c_3r_1 & c_1p_2 + c_2q_2 + c_3r_2 & c_1p_3 + c_2q_3 + c_3r_3 \end{bmatrix}. \end{aligned}}$$

- **Break-down:**

A and B are both 3×3 matrices $\Longrightarrow$ AB is a 3×3 matrix.

In other words:

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix} = \begin{bmatrix} \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix}.$$

(i) Let us find $\diamondsuit$ in

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{\quad\diamondsuit\quad} & \boxed{\qquad\qquad} & \boxed{\qquad\qquad} \\ \boxed{\qquad\qquad} & \boxed{\qquad\qquad} & \boxed{\qquad\qquad} \\ \boxed{\qquad\qquad} & \boxed{\qquad\qquad} & \boxed{\qquad\qquad} \end{bmatrix}.$$

Since $\diamondsuit$ is in the top-left, accordingly highlight the portion of A and B, like

$$\begin{bmatrix} \boxed{a_1 \quad a_2 \quad a_3} \\ b_1 \quad b_2 \quad b_3 \\ c_1 \quad c_2 \quad c_3 \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} p_1 \\ q_1 \\ r_1 \end{matrix}} & \begin{matrix} p_2 \\ q_2 \\ r_2 \end{matrix} & \begin{matrix} p_3 \\ q_3 \\ r_3 \end{matrix} \end{bmatrix}.$$

$\diamondsuit$ is $a_1p_1 + a_2q_1 + a_3r_1$:

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{a_1p_1 + a_2q_1 + a_3r_1} & \boxed{\qquad\qquad} & \boxed{\qquad\qquad} \\ \boxed{\qquad\qquad} & \boxed{\qquad\qquad} & \boxed{\qquad\qquad} \\ \boxed{\qquad\qquad} & \boxed{\qquad\qquad} & \boxed{\qquad\qquad} \end{bmatrix}.$$

(ii) Next, let's find $\heartsuit$ in

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{a_1p_1 + a_2q_1 + a_3r_1} & \boxed{\heartsuit} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix}$$

Since $\heartsuit$ is the top-middle $\big($top-row & middle-column$\big)$, accordingly highlight the portion of A and B, like

$$\begin{bmatrix} \boxed{a_1 \quad a_2 \quad a_3} \\ b_1 \quad b_2 \quad b_3 \\ c_1 \quad c_2 \quad c_3 \end{bmatrix} \begin{bmatrix} p_1 & \boxed{\begin{matrix} p_2 \\ q_2 \\ r_2 \end{matrix}} & p_3 \\ q_1 & & q_3 \\ r_1 & & r_3 \end{bmatrix}.$$

$\heartsuit$ is $a_1p_2 + a_2q_2 + a_3r_2$:

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{a_1p_1 + a_2q_1 + a_3r_1} & \boxed{a_1p_2 + a_2q_2 + a_3r_2} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix}.$$

(iii) Similarly, we can find $\clubsuit$ in

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{a_1p_1 + a_2q_1 + a_3r_1} & \boxed{a_1p_2 + a_2q_2 + a_3r_2} & \boxed{\clubsuit} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix}$$

as

$$\begin{bmatrix} \boxed{a_1 \quad a_2 \quad a_3} \\ b_1 \quad b_2 \quad b_3 \\ c_1 \quad c_2 \quad c_3 \end{bmatrix} \begin{bmatrix} p_1 & p_2 & \begin{bmatrix} p_3 \\ q_3 \\ r_3 \end{bmatrix} \\ q_1 & q_2 & \\ r_1 & r_2 & \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{a_1p_1 + a_2q_1 + a_3r_1} & \boxed{a_1p_2 + a_2q_2 + a_3r_2} & \boxed{a_1p_3 + a_2q_3 + a_3r_3} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix}.$$

(iv) Next, we can find $\spadesuit$ in

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{a_1p_1 + a_2q_1 + a_3r_1} & \boxed{a_1p_2 + a_2q_2 + a_3r_2} & \boxed{a_1p_3 + a_2q_3 + a_3r_3} \\ \boxed{\spadesuit} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix}$$

as

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ \boxed{b_1 \quad b_2 \quad b_3} \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} p_1 \\ q_1 \\ r_1 \end{matrix}} & \begin{matrix} p_2 \\ q_2 \\ r_2 \end{matrix} & \begin{matrix} p_3 \\ q_3 \\ r_3 \end{matrix} \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{a_1p_1 + a_2q_1 + a_3r_1} & \boxed{a_1p_2 + a_2q_2 + a_3r_2} & \boxed{a_1p_3 + a_2q_3 + a_3r_3} \\ \boxed{b_1p_1 + b_2q_1 + b_3r_1} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix}.$$

Now, the rest goes the same way.

Example. For $A = \begin{bmatrix} 1 & -1 & 7 \\ 2 & -1 & 8 \\ 3 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 2 \end{bmatrix}$,

$$\begin{aligned}
AB &= \begin{bmatrix} 1 & -1 & 7 \\ 2 & -1 & 8 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 1\cdot1 + (-1)\cdot2 + 7\cdot1 & 1\cdot1 + (-1)\cdot1 + 7\cdot(-3) & 1\cdot2 + (-1)\cdot1 + 7\cdot2 \\ 2\cdot1 + (-1)\cdot2 + 8\cdot1 & 2\cdot1 + (-1)\cdot1 + 8\cdot(-3) & 2\cdot2 + (-1)\cdot1 + 8\cdot2 \\ 3\cdot1 + 1\cdot2 + (-1)\cdot1 & 3\cdot1 + 1\cdot1 + (-1)\cdot(-3) & 3\cdot2 + 1\cdot1 + (-1)\cdot2 \end{bmatrix} \\
&= \begin{bmatrix} 6 & -21 & 15 \\ 8 & -23 & 19 \\ 4 & 7 & 5 \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
BA &= \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 7 \\ 2 & -1 & 8 \\ 3 & 1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1\cdot1 + 1\cdot2 + 2\cdot3 & 1\cdot(-1) + 1\cdot(-1) + 2\cdot1 & 1\cdot7 + 1\cdot8 + 2\cdot(-1) \\ 2\cdot1 + 1\cdot2 + 1\cdot3 & 2\cdot(-1) + 1\cdot(-1) + 1\cdot1 & 2\cdot7 + 1\cdot8 + 1\cdot(-1) \\ 1\cdot1 + (-3)\cdot2 + 2\cdot3 & 1\cdot(-1) + (-3)\cdot(-1) + 2\cdot1 & 1\cdot7 + (-3)\cdot8 + 2\cdot(-1) \end{bmatrix} \\
&= \begin{bmatrix} 9 & 0 & 13 \\ 7 & -2 & 21 \\ 1 & 4 & -19 \end{bmatrix}.
\end{aligned}$$

So, once again, $\big($just like the 2×2 case$\big)$ $\underline{\underline{\text{in general, } AB \text{ and } BA \text{ are not equal.}}}$

Exercise (= "VIII"; Exercise 2). Calculate AB and BA:

(1) $A = \begin{bmatrix} 2 & 1 & 3 \\ -2 & 2 & 3 \\ 0 & -1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 & 2 \\ 1 & 3 & 1 \\ -1 & 2 & -1 \end{bmatrix}.$

(2) $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$

(3) $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -4 & 0 \\ -1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}.$

(4) $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$

- **Definition (Scalar multiplied to a matrix).** Let s be a scalar. Then

$$\boxed{s \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} sa_1 & sa_2 & sa_3 \\ sb_1 & sb_2 & sb_3 \\ sc_1 & sc_2 & sc_3 \end{bmatrix}.}$$

Paraphrase:

$$\boxed{\begin{array}{l} \text{If} \quad A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad \text{and} \quad s \,:\, \text{a scalar} \\ \qquad \Longrightarrow \qquad sA = \begin{bmatrix} sa_1 & sa_2 & sa_3 \\ sb_1 & sb_2 & sb_3 \\ sc_1 & sc_2 & sc_3 \end{bmatrix}. \end{array}}$$

Inverse of a 3×3 matrix.

Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$. The $\underline{\text{ inverse }}$ A^{-1} of A is the following matrix:

$$\boxed{A^{-1} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}^{-1} = \frac{1}{\det A} \operatorname{adj} A,}$$

where

$$\det A = a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1,$$

and

$$\operatorname{adj} A = \begin{bmatrix} + \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} & - \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} & + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \\ - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} & + \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} & - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \\ + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} & - \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} & + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \end{bmatrix}.$$

A^{-1} exists, provided $\det A \neq 0$.

Example. For $A = \begin{bmatrix} 2 & 1 & -2 \\ 5 & -4 & -1 \\ 1 & -3 & 4 \end{bmatrix}$, find A^{-1}.

Step 1. Find the determinant of A:

$$\begin{aligned}
\det A &= 2 \cdot \begin{vmatrix} -4 & -1 \\ -3 & 4 \end{vmatrix} - 1 \cdot \begin{vmatrix} 5 & -1 \\ 1 & 4 \end{vmatrix} + \left(-2\right) \cdot \begin{vmatrix} 5 & -4 \\ 1 & -3 \end{vmatrix} \\
&= 2 \cdot \left(-19\right) - 1 \cdot \left(-21\right) + \left(-2\right) \cdot \left(-11\right) \\
&= -38 - 21 + 22 = -37.
\end{aligned}$$

Step 2. Find $\operatorname{adj} A$:

$$\operatorname{adj} A = \begin{bmatrix} + \begin{vmatrix} -4 & -1 \\ -3 & 4 \end{vmatrix} & - \begin{vmatrix} 1 & -2 \\ -3 & 4 \end{vmatrix} & + \begin{vmatrix} 1 & -2 \\ -4 & -1 \end{vmatrix} \\ - \begin{vmatrix} 5 & -1 \\ 1 & 4 \end{vmatrix} & + \begin{vmatrix} 2 & -2 \\ 1 & 4 \end{vmatrix} & - \begin{vmatrix} 2 & -2 \\ 5 & -1 \end{vmatrix} \\ + \begin{vmatrix} 5 & -4 \\ 1 & -3 \end{vmatrix} & - \begin{vmatrix} 2 & 1 \\ 1 & -3 \end{vmatrix} & + \begin{vmatrix} 2 & 1 \\ 5 & -4 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -19 & 2 & -9 \\ -21 & 10 & -8 \\ -11 & 7 & -13 \end{bmatrix}.$$

To conclude,

$$A^{-1} = \frac{1}{-37} \begin{bmatrix} -19 & 2 & -9 \\ -21 & 10 & -8 \\ -11 & 7 & -13 \end{bmatrix}$$

$$= \frac{1}{37} \begin{bmatrix} 19 & -2 & 9 \\ 21 & -10 & 8 \\ 11 & -7 & 13 \end{bmatrix}$$

$$\left(= \begin{bmatrix} \frac{19}{37} & \frac{-2}{37} & \frac{9}{37} \\ \frac{21}{37} & \frac{-10}{37} & \frac{8}{37} \\ \frac{11}{37} & \frac{-7}{37} & \frac{13}{37} \end{bmatrix} \right).$$

Example. For $A = \begin{bmatrix} 1 & -3 & 2 \\ 3 & -5 & 2 \\ 6 & -6 & 2 \end{bmatrix}$, find A^{-1}.

Step 1. Find the determinant of A:

$$\begin{aligned} \det A &= 1 \cdot \begin{vmatrix} -5 & 2 \\ -6 & 2 \end{vmatrix} - \big(-3\big) \cdot \begin{vmatrix} 3 & 2 \\ 6 & 2 \end{vmatrix} + 2 \cdot \begin{vmatrix} 3 & -5 \\ 6 & -6 \end{vmatrix} \\ &= 1 \cdot 2 - \big(-3\big) \cdot \big(-6\big) + 2 \cdot 12 \\ &= 2 - 18 + 24 = 8. \end{aligned}$$

Step 2.

$$\operatorname{adj} A = \begin{bmatrix} +\begin{vmatrix} -5 & 2 \\ -6 & 2 \end{vmatrix} & -\begin{vmatrix} -3 & 2 \\ -6 & 2 \end{vmatrix} & +\begin{vmatrix} -3 & 2 \\ -5 & 2 \end{vmatrix} \\ -\begin{vmatrix} 3 & 2 \\ 6 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 6 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} \\ +\begin{vmatrix} 3 & -5 \\ 6 & -6 \end{vmatrix} & -\begin{vmatrix} 1 & -3 \\ 6 & -6 \end{vmatrix} & +\begin{vmatrix} 1 & -3 \\ 3 & -5 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -6 & 4 \\ 6 & -10 & 4 \\ 12 & -12 & 4 \end{bmatrix}.$$

To conclude,

$$\begin{aligned} A^{-1} &= \frac{1}{8} \begin{bmatrix} 2 & -6 & 4 \\ 6 & -10 & 4 \\ 12 & -12 & 4 \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} 1 & -3 & 2 \\ 3 & -5 & 2 \\ 6 & -6 & 2 \end{bmatrix} \left(= \begin{bmatrix} \frac{1}{4} & \frac{-3}{4} & \frac{1}{2} \\ \frac{3}{4} & \frac{-5}{4} & \frac{1}{2} \\ \frac{3}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} \right). \end{aligned}$$

- **The 3×3 identity matrix.**

Define

$$\boxed{I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.}$$

We call it the 3×3 identity matrix. If you want to be meticuous, you can denote it I_3 to indicate the size.

Fact 1. For $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, we have

$$\boxed{IA = A, \quad \text{and} \quad AI = A.}$$

Fact 2. For $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, suppose

$$\det A \neq 0.$$

Then

$$\boxed{AA^{-1} = I, \quad \text{and} \quad A^{-1}A = I.}$$

Exercise (= "VIII"; Exercise 3). Prove Fact 1 and Fact 2 above (brute-force).

- **Gaussian elimination.**

Example. Consider

$$\begin{cases} x + y + z = 2, \\ -x + 3y + 2z = 8, \\ 4x + y = 4. \end{cases}$$

Solve this system <u>brute-force, step-by-step</u>.

Step 1. Multiply 2 to the first equation in the system sidewise. The result is

$$2x + 2y + 2z = 4.$$

Step 2. Subtract it from the second equation in the given system sidewise. The result is

$$-3x + y = 4.$$

Step 3. Subtract it from the third equation in the given system sidewise. The result is

$$7x = 0.$$

Step 4. Multiply $\frac{1}{7}$ to the two sides. The result is

$$x = 0.$$

Step 5. Go back to Step 2:

$$-3x + y = 4.$$

Substitute the outcome of Step 4: $x = 0$. The result is

$$y = 4.$$

Step 6. Go back to the first equation in the original system:

$$x + y + z = 2.$$

Substitute the outcomes of Step 4 and Step 5: $x = 0$, $y = 4$. The result is

$$4 + z \;=\; 2.$$

Solve it for z:

$$z \;=\; -2.$$

In sum, we have obtained the solution

$$\Big(x,\ y,\ z\Big) \;=\; \Big(0,\ 4,\ -2\Big).$$

§9. Gaussian Elimination.

The above method is called "Gaussian elimination". Below we frame it in the context of matrix operations.

Problem 1. Solve the following system $\big($the same as above$\big)$

$$\begin{cases} x + y + z \;=\; 2, \\ -x + 3y + 2z \;=\; 8, \\ 4x + y \;=\; 4 \end{cases}$$

using matrices.

Solution using matrices. Construct the so-called augmented matrix

$$(*) \qquad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ -1 & 3 & 2 & 8 \\ 4 & 1 & 0 & 4 \end{array}\right].$$

The vertical rule actually doesn't play any role. So you won't see it below.

Goal. Re-enact the steps as above, using matrices, and ultimately reduce $(*)$ to

$$(\#) \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

Step 1.

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ -1 & 3 & 2 & 8 \\ 4 & 1 & 0 & 4 \end{bmatrix} \begin{array}{l} \cdot(-2) \\ \\ \\ \end{array} \quad \text{add up}$$

$$\to \begin{bmatrix} 1 & 1 & 1 & 2 \\ -3 & 1 & 0 & 4 \\ 4 & 1 & 0 & 4 \end{bmatrix}$$

(Keep the original top row intact.)

Step 2.

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ -3 & 1 & 0 & 4 \\ 4 & 1 & 0 & 4 \end{bmatrix} \begin{array}{l} \\ \cdot(-1) \\ \\ \end{array} \quad \text{add up}$$

$$\to \begin{bmatrix} 1 & 1 & 1 & 2 \\ -3 & 1 & 0 & 4 \\ 7 & 0 & 0 & 0 \end{bmatrix}$$

(Keep the original middle row intact.)

Step 3.

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ -3 & 1 & 0 & 4 \\ 7 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{multiply } \tfrac{1}{7}$$

$$\to \begin{bmatrix} 1 & 1 & 1 & 2 \\ -3 & 1 & 0 & 4 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Step 4.

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ -3 & 1 & 0 & 4 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{interchange}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 4 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Step 5.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 4 \\ 1 & 1 & 1 & 2 \end{bmatrix} \begin{array}{l} \cdot 3 \\ \\ \\ \end{array} \quad \text{add up}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

(Keep the original top row intact.)

Step 6.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 1 & 1 & 1 & 2 \end{bmatrix} \begin{array}{l} \cdot (-1) \\ \\ \\ \end{array} \quad \text{add up}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

(Keep the original top row intact.)

Step 7.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \boxed{0 \quad 1 \quad 0 \quad 4} \\ \boxed{0 \quad 1 \quad 1 \quad 2} \end{bmatrix} \begin{matrix} \\ \cdot\,(-1) \\ \\ \end{matrix} \quad \text{add up}$$

$$\to \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ \boxed{0 \quad 0 \quad 1 \quad -2} \end{bmatrix}$$

(Keep the original middle row intact.)

This last matrix is of the form

$$(\#) \qquad \left[\begin{array}{c|c} \underbrace{\boxed{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}}_{\| \atop I} & \begin{matrix} 0 \\ 4 \\ -2 \end{matrix} \end{array}\right] \quad .$$

The boxed part is the identity matrix I. So, the answer (namely, the x-, y- and z-values) is found in the rigt-most column:

$$\left(x, \quad y, \quad z\right) = \left(0, \quad 4, \quad -2\right).$$

Question 1a. What sort of operations are allowed at every step of the reduction process $(*) \to (\#)$?

Answer to Question 1a.

Three operations, called elementary row operations, are allowed:

- multiply a scalar t to one entire row, where $t \neq 0$.
- add [t (a scalar) times row #a] to row #b (while keeping row #a intact).
- Interchange row #a and row #b.

Question 1b. Is there a name for something like $(\#)$ $\big($in the previous page$\big)$?

Answer to Question 1b.

Yes: Reduced row echelon form. More precisely, any 3×4 matrix whose shape falls into one of the following fifteen types is called a reduced row echelon form, where each of the '$*$' spots is filled by an arbitrary number:

$$\begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & * & 0 & 0 \\ 0 & * & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

- Any reduced echelon form, in any size $\big($not just $3 \times 4\big)$, obeys the following:

(i) If you read off each row from left to right, then either it starts with 1, or it starts with 0 and 0 repeats until 1 shows up at some point, or the whole row is entirely 0. The first 1 from the left, if any, is called the leading 1.

(ii) A column that contains a leading 1 has 0 everywhere else.

(iii) By (ii), if there are two leading 1s, they cannot coexist in the same column. In that case the lower one sits further right to the upper one.

(iv) A row that consists entirely 0, if any, are grouped together at the bottom.

The properties (i), (ii), (iii) and (iv) characterize reduced echelon forms in any size.

Example. The following 3×4 matrices are in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Exercise (= "IX"; Exercise 1). Which one of the following are in reduced row echelon form?

(1) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, (2) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, (3) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

(4) $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, (5) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \end{bmatrix}$, (6) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$,

(7) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, (8) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, (9) $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, (10) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

(11) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, (12) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, (13) $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$.

Exercise (= "IX"; Exercise 2). List up all possible 3×6 reduced row echelon forms. There are forty two (42) different types.

Question 1c. Is it always feasible to reduce something like $(*)$ to something like

(#) $$\left[\begin{array}{c} \boxed{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}} \\ \| \\ I \end{array} \begin{matrix} * \\ * \\ * \end{matrix}\right].$$

Answer to Question 1c.

No, it is not always true that any augmented 3×4 matrix can be reduced to (#).

However, it is true that any augmented 3×4 matrix can be reduced to a reduced row echelon form $\big($either one of the fifteen in the list in page 7$\big)$. More generally, it is true that any matrix can be reduced to a reduced row echelon form of the same size via elementary row operations.

Question 2a. Basically, the system in Problem 1 is of the form $A\boldsymbol{x} = \boldsymbol{b}$. We already know how to invert A. So, the "golden rule"

$$\boxed{\quad A\boldsymbol{x} = \boldsymbol{b} \quad \underset{\substack{\text{can solve,} \\ \text{if } \det A \neq 0}}{\Longrightarrow} \quad \boldsymbol{x} = A^{-1}\boldsymbol{b} \quad}$$

should take care of the problem. Then isn't the above 'solution' redundant?

Question 2b. That said, computation of A^{-1} is very cumbersome. So, basically the above 'solution' replaces the "golden rule". Then what is the use of the inverse of a matrix?

Question 3. Back to Question 2a, what happens if A^{-1} does not exist? What can one say about the root of the system?

Answer to Questions 2a & 3.

It is absolutely correct, that the "golden rule" applies to Problem 1. The above solution gives an alternative way to pull the answer. That said, the 'golden rule' works only when $\det A \neq 0$. In the above, we didn't know whether $\det A \neq 0$ beforehand. The above method at our disposal even when we don't know whether $\det A \neq 0$ beforehand.

Answer to Question 2b.

Now, what I just said might suggest that the notion of the inverse of matrices is redundant. The truth is, the exact same method — to reduce a given matrix to a reduced row echelon form — can be applied to calculate A^{-1} of a given matrix A. The gist of what we have worked out above essentially amounts to calculating A^{-1}.

Formula 1. For a 3×3 matrix $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, construct

$$\left[\, A \,\middle|\, I \,\right] = \begin{bmatrix} a_1 & a_2 & a_3 & 1 & 0 & 0 \\ b_1 & b_2 & b_3 & 0 & 1 & 0 \\ c_1 & c_2 & c_3 & 0 & 0 & 1 \end{bmatrix}.$$

This is a 3×6 matrix.

(1) If the reduced row echelon form of $\left[\, A \,\middle|\, I \,\right]$ is of the form

$$\left[\, I \,\middle|\, B \,\right] = \begin{bmatrix} 1 & 0 & 0 & p_1 & p_2 & p_3 \\ 0 & 1 & 0 & q_1 & q_2 & q_3 \\ 0 & 0 & 1 & r_1 & r_2 & r_3 \end{bmatrix},$$

then $\det A \neq 0$, and moreover

$$B = \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix}$$

is the inverse of A: $B = A^{-1}$.

(2) If the reduced row echelon form of $\left[\, A \,\middle|\, I \,\right]$ is of the form

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * \end{bmatrix},$$

then $\det A = 0$. In this case, A^{-1} does not exist.

Example. Find A^{-1} for $A = \begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$, if exists.

Step 1. Form

$$\left[\, A \,\middle|\, I \,\right] = \begin{bmatrix} -2 & 2 & 3 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 0 & 0 & 1 \end{bmatrix}.$$

Step 2. Apply Gaussian elimination method to reduce this matrix to a reduced row echelon form $\big($in what follows, top row; middle row, and bottom row, will be referred to as $\big($row 1$\big)$; $\big($row 2$\big)$, and $\big($row 3$\big)$, respectively$\big)$.

$$\begin{bmatrix} -2 & 2 & 3 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & -3/2 & -1/2 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 0 & 0 & 1 \end{bmatrix}$$

$\Big[\ \big(-1/2\big)$ was multiplied to $\big($row 1$\big)\ \Big]$

$$\rightarrow \begin{bmatrix} 1 & -1 & -3/2 & -1/2 & 0 & 0 \\ 0 & 0 & 3/2 & 1/2 & 1 & 0 \\ 0 & 1 & 4 & 0 & 0 & 1 \end{bmatrix}$$

$\Big[\ \big(-1\big)$ times $\big($row 1$\big)$ was added to $\big($row 2$\big)\ \Big]$

$$\rightarrow \begin{bmatrix} 1 & -1 & -3/2 & -1/2 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 & 1 \\ 0 & 0 & 3/2 & 1/2 & 1 & 0 \end{bmatrix}$$

$\Big[\ \big($row 2$\big)$ and $\big($row 3$\big)$ were interchanged $\Big]$

$$\rightarrow \begin{bmatrix} 1 & 0 & 5/2 & -1/2 & 0 & 1 \\ 0 & 1 & 4 & 0 & 0 & 1 \\ 0 & 0 & 3/2 & 1/2 & 1 & 0 \end{bmatrix}$$

$\Big[\ \big($row 2$\big)$ was added to $\big($row 1$\big)\ \Big]$

$$\rightarrow \begin{bmatrix} 1 & 0 & 5/2 & -1/2 & 0 & 1 \\ 0 & 1 & 4 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1/3 & 2/3 & 0 \end{bmatrix}$$

$\Big[\ \big(2/3\big)$ was multiplied to $\big($row 3$\big)\ \Big]$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & -4/3 & -5/3 & 1 \\ 0 & 1 & 0 & -4/3 & -8/3 & 1 \\ 0 & 0 & 1 & 1/3 & 2/3 & 0 \end{bmatrix}$$

$\Big[\ \big(-5/2\big)$ times $\big($row 3$\big)$ was added to $\big($row 1$\big)$;

$\big(-4\big)$ times $\big($row 3$\big)$ was added to $\big($row 2$\big)\ \Big]$.

So, by part (1) of Formula 1, A^{-1} indeed exists, and it is

$$A^{-1} = \begin{bmatrix} -4/3 & -5/3 & 1 \\ -4/3 & -8/3 & 1 \\ 1/3 & 2/3 & 0 \end{bmatrix}.$$

Exercise (= "IX"; Exercise 3). Verify that the result for A^{-1} in Example above is correct, by way of physically calculating $A^{-1}A$ $\big($or alternatively, $AA^{-1}\big)$. If the outcome equals I, then this answer is indeed correct. $\big($The proof of the fact that this is indeed the correct checking method is pending.$\big)$

Example 3. Find B^{-1} for $B = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -10 \\ 7 & 16 & -21 \end{bmatrix}$, if exists.

Step 1. Form

$$\Big[\, B \,\Big|\, I \,\Big] = \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 7 & -10 & 0 & 1 & 0 \\ 7 & 16 & -21 & 0 & 0 & 1 \end{bmatrix}.$$

Step 2. Apply Gaussian elimination method to reduce this matrix to a reduced row echelon form $\big($in what follows, top row; middle row, and bottom row, will be referred to as $\big($row 1$\big)$; $\big($row 2$\big)$, and $\big($row 3$\big)$, respectively$\big)$.

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 7 & -10 & 0 & 1 & 0 \\ 7 & 16 & -21 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -7 & -3 & 1 & 0 \\ 0 & 2 & -14 & -7 & 0 & 1 \end{bmatrix}$$

$$\Big[\ \big(-3\big)\ \text{times}\ \big(\text{row } 1\big)\ \text{was added to}\ \big(\text{row } 2\big);$$

$$\big(-7\big)\ \text{times}\ \big(\text{row } 1\big)\ \text{was added to}\ \big(\text{row } 3\big)\ \Big]$$

$$\to \quad \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -7 & -3 & 1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 \end{bmatrix}.$$

$$\Big[\ \big(-2\big)\ \text{times}\ \big(\text{row } 2\big)\ \text{was added to}\ \big(\text{row } 3\big)\ \Big]$$

This last matrix is of the form

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * \end{bmatrix}.$$

So, by part (2) of Formula 1, B^{-1} does not exist.

Exercise (= "IX"; Exercise 4). For B as above, verify that B^{-1} does not exist, independently of the above, by way of calculating $\det B$. If the outcome equals 0, then the above conclusion, that B^{-1} does not exist, is indeed correct.

Exercise (= "IX"; Exercise 5). Use Formula 1 to invert each of the six matrices (1–6) in Exercise 5, page 14 of "Review of Lectures – III". Verify that you get the same answer for each of (1–6). Let me duplicate them below:

(1) $\quad A = \begin{bmatrix} 2 & 1 & -2 \\ 5 & -4 & -1 \\ 1 & -3 & 4 \end{bmatrix}.$ $\qquad$ (2) $\quad A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 1 \\ 1 & -2 & -2 \end{bmatrix}.$

(3) $\quad A = \begin{bmatrix} 3 & 4 & -4 \\ 2 & 1 & 4 \\ -2 & 4 & 1 \end{bmatrix}.$ $\qquad$ (4) $\quad A = \begin{bmatrix} 3 & 5 & 10 \\ 3 & 1 & 6 \\ -2 & -2 & -6 \end{bmatrix}.$

$$(5) \qquad A = \begin{bmatrix} 1 & \sqrt{2} & \sqrt{3} \\ \sqrt{2} & \dfrac{-2-3\sqrt{6}}{5} & \dfrac{6-\sqrt{6}}{5} \\ \sqrt{3} & \dfrac{6-\sqrt{6}}{5} & \dfrac{-3-2\sqrt{6}}{5} \end{bmatrix}.$$

$$(6) \qquad A = \begin{bmatrix} \dfrac{2+3\sqrt{2}}{8} & \dfrac{-2\sqrt{3}+\sqrt{6}}{8} & \dfrac{\sqrt{6}}{4} \\ \dfrac{-2\sqrt{3}+\sqrt{6}}{8} & \dfrac{6+\sqrt{2}}{8} & \dfrac{\sqrt{2}}{4} \\ -\dfrac{\sqrt{6}}{4} & -\dfrac{\sqrt{2}}{4} & \dfrac{\sqrt{2}}{2} \end{bmatrix}.$$